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# REAL REPRESENTATIONS OF GROUPS WITH A SINGLE INVOLUTION

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**If $G$ is a finite group containing just one involution and $G$ has a faithful, absolutely irreducible real representation, then $G$ has order 2.**

This was proved by Jerry Malzan [2] using the classification of simple groups with dihedral Sylow 2-subgroups. The purpose of this note is to give a proof of Malzan's theorem which assumes nothing but some elementary character theory.

Let $G$ have the unique involution $z$ and assume $G > \langle z \rangle$. Let $\chi \in \mathrm{Irr}(G)$ be faithful and real valued (where $\mathrm{Irr}(G)$ is the set of complex irreducible characters of $G$). By the Frobenius-Schur theory (see Lemma 4.4 and Corollary 4.15 of [1]) it follows that in order to prove that $\chi$ is not afforded by a real representation, it suffices to show that

$$\sum_{g \in G} \chi(g^2) \neq |G| .$$

THEOREM. *In the above situation we have*

$$\sum_{g \in G} \chi(g^2) < |G| .$$

*Proof*. Each $g \in G$ may be uniquely factored as $g = \sigma c$ where $\sigma$ has 2-power order and $c \in C(\sigma)$ has odd order. We write $\sigma = g_2$. For each cyclic 2-subgroup $U \subseteq G$ we set $Y(U) = \{g \in G \mid \langle g_2 \rangle = U\}$. Thus the sets $Y(U)$ partition $G$. We shall prove

$$(1) \qquad \sum_{g \in Y(1)} \chi(g^2) = \sum_{g \in Y(\langle z \rangle)} \chi(g^2) < |G|/2$$

$$(2) \qquad \sum_{g \in Y(U)} \chi(g^2) \leqq 0 \quad \text{if} \quad |U| = 4$$

$$(3) \qquad \sum_{g \in Y(U)} \chi(g^2) = 0 \quad \text{if} \quad |U| \geqq 8 .$$

The theorem will then follow.

*Proof of* (1). $Y(1)$ is the set of elements of $G$ of odd order and since $z \in \mathbb{Z}(G)$, we have $Y(\langle z \rangle) = zY(1)$ and so $\sum_{Y(1)} \chi(g^2) = \sum_{Y(\langle z \rangle)} \chi(g^2)$. Since the map $g \mapsto g^2$ is a permutation of $Y(1)$, the common value of these sums is

$$s = \sum_{g \in Y(1)} \chi(g) .$$