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ON AN ADDITIVE ARITHMETIC FUNCTION

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We discuss in this paper arithmetic properties of the function $A(n) = \sum_{p^\alpha || n} \alpha p$. Asymptotic estimates of $A(n)$ reveal the connection between $A(n)$ and large prime factors of n. The distribution modulo 2 of $A(n)$ turns out to be an interesting study and congruences involving $A(n)$ are considered. Moreover the very intimate connection between $A(n)$ and the partition of integers into primes provides a natural motivation for its study.

0. **Introduction.** Let a positive integer n be expressed as a product of distinct primes in the canonical fashion $n = \prod_{i=1}^{r} p_i^{\alpha_i}$. Define a function $A(n) = \sum_{i=1}^{r} \alpha_i p_i$.

(i) The function $A(n)$ is not injective. In fact for a fixed integer m, the number of solutions in n to $A(n) = m$, is the number of partitions of m into primes.

(ii) $A(n)$ fluctuates in size appreciably. It is easily seen that $A(n) = n$ when n is a prime, while $A(n) = O(\log n)$ when n is a power of a small prime. Actually the "average order" of $A(n)$ turns out (as a corollary to Theorem 1.1) to be $\pi^2 n/6 \log n$. The term average order is defined below.

(iii) The function $A(n)$ is additive and one can expect it to take odd and even values with equal frequency.

The term "average order" calls for some explanation. We follow the usage in Hardy and Wright [6]. If $f(n)$ is a function defined on the positive integers we consider

$$F(x) = \sum_{n \leq x} f(n) .$$

Usually F can be expressed in terms of well behaved functions like polynomials or exponentials and the like. That is we seek an asymptotic estimate for F in terms of these functions. Then we seek a similar well behaved function g so that

$$F(x) = \sum_{n \leq x} f(n) \sim \sum_{n \leq x} g(n) .$$

The function g may be thought of as the average order of f. For instance if φ is the Euler function then

$$F(x) = \sum_{n \leq x} \varphi(n) = \frac{3x^2}{\pi^2} + O(x \log x) \sim \sum_{n \leq x} \frac{6n}{\pi^2}$$

so the average order of $\varphi(n)$ is $6n/\pi^2$.