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A NOTE ON THE JACOBI-PERRON ALGORITHM

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In this paper we give a simple geometric description of the Jacobi-Perron algorithm, based on the matrix-theoretic approach to the algorithm. An important advantage of the geometric description is that it may be used as an aid to intuition as well as a practical tool. As an illustration, we prove convergence for a special case.

The theory and procedures of the JPA have been studied from many different viewpoints (cf. [1] and [3]). We consider here some of the linear algebra and geometry naturally associated with the matrix theory of the JPA.

In particular, for $n = 2, 3$ the procedures of the JPA may be represented concretely in the Euclidean spaces R^2 and R^3, and the notion of convergence of the JPA takes on a fairly simple geometrical meaning.

In § 1 we summarize briefly the JPA as described by Bernstein in [1] (throughout, we use [1] as the standard reference). In § 2, we restate the definitions given in § 1 in matrix-theoretic terms. Based on this, we give a general description of the geometrical meaning which may be attached to the notion of convergence of the JPA. In § 3 we consider in detail the geometry associated with the JPA for $n = 3$. The behavior of a JPA may be represented graphically in this case. (This may be done similarly for $n = 2$, but the case $n = 3$ is far more representative of the general case.) We conclude with a straightforward, elementary proof of the convergence of any JPA for a T-function whose values are positive integers, for the case $n = 3$.

1. The Jacobi-Perron algorithm (JPA).

In this section, we briefly recall the description of the JPA given in [1], and throughout this section we use the notation of [1].

Let n be fixed, and let k be a nonnegative integer. The vector $a^{(k)}$ in R^{n-1} is defined by

$$a^{(k)} = (a_1^{(k)}, a_2^{(k)}, \cdots, a_{n-1}^{(k)}) . \tag{1.1}$$

A transformation T of R^{n-1} to itself is defined as follows: suppose f is a (vector) function on R^{n-1} such that

$$f(a^{(k)}) = b^{(k)} = (b_1^{(k)}, b_2^{(k)}, \cdots, b_{n-1}^{(k)}) \tag{1.2}$$

and suppose also that $a_1^{(k)} \neq b_1^{(k)}$. Then put