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MONOTONICITY AND ALTERNATIVE METHODS FOR NONLINEAR BOUNDARY VALUE PROBLEMS

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Let X be a Hilbert space, E a linear operator with finite dimensional null space, and N a nonlinear operator. In this paper we study the nonlinear equation

$$Ex = Nx \quad x \in X . \tag{1}$$

Equations of this form arise in the study of boundary value problems for elliptic differential equations.

We use the alternative scheme of Bancroft, Hale, and Sweet and results from monotone operator theory with suitable monotonicity assumptions on E and N to reduce equation (1) to an alternative problem. We then use results from monotone operator theory to solve the alternative problem, hence prove the existence of solutions to equation (1). This extends to nonselfadjoint operators the results of Cesari and Kannan.

1. Introduction. The reduction of equation (1) to a finite dimensional alternative problem has been done using the contraction mapping principle by Cesari [4] for selfadjoint operators and by Bancroft, Hale, and Sweet [1] for nonselfadjoint operators; and using the theory of monotone operators by Gustafson and Sather [8] for selfadjoint operators where E may have continuous spectrum, by Cesari and Kannan [7] for selfadjoint operators with a complete set of eigenfunctions and eigenvalues approaching $-\infty$, by Cesari [6] for nonselfadjoint operators, and by Osborn and Sather [11] for nonselfadjoint operators generated by a coercive bilinear form. Only the papers of Gustafson and Sather [8] and Cesari [6] avoid using a compactness argument such as assuming that E has a compact resolvent $(E - aI)^{-1}$. For a survey of recent results see Cesari [5] for selfadjoint problems and Cesari [6] for nonselfadjoint problems.

Since the alternative problem is now on a finite dimensional subspace of X, it has been the practice to use either degree theory or the implicit function theorem to solve the alternative problem. An exception to this is the paper by Cesari and Kannan [7] which uses monotone operator theory to solve the alternative problem hence obtain a solution to equation (1).

In § 3 of this paper we use the alternative scheme of Bancroft, Hale, and Sweet [1] but with the theory of monotone operators to reduce equation (1) to a finite dimensional alternative problem (Theorem 4). This reduction is a modification of the method used