PACIFIC JOURNAL OF MATHEMATICS
Vol. 72, No. 2, 1977

W_δ (T) IS CONVEX

J. KYLE

Stampfli introduced a generalization of the numerical range for any bounded linear operator T on a Hilbert space $\mathscr{H}$. This is denoted by $W_\delta(T)$ and is defined by

$$W_\delta(T) = \textbf{closure } \{\langle Tx, x\rangle : ||\mathbf{x}|| = 1 \textbf{ and } ||Tx|| \geqq \delta\} .$$

Stampfli asked whether $W_\delta(T)$ is convex. In this short note we provide an affirmative answer to this question.

$\mathscr{L}(\mathscr{H})$ will denote the set of bounded linear operators on the Hilbert space $\mathscr{H}$.

LEMMA 1. *Suppose S and A belong to $\mathscr{L}(\mathscr{H})$, and that $S = S^*$. Then*

$$S(A, \delta) = \{x \in \mathscr{H} : ||x|| = 1 \quad \text{and} \quad ||Ax|| \geqq \delta \quad \text{and} \quad \langle Sx, x\rangle = 0\}$$

is path connected.

Proof. Suppose x and y belong to $S(A, \delta)$. We may assume that x and y are linearly independent. (If not, they both lie on an arc of

$$\{e^{i\theta}x : 0 \leqq \theta \leqq 2\pi\}$$

which lies in $S(A, \delta)$ if x does.)

Choose θ in $\boldsymbol{R}$ such that $e^{i\theta}\langle Sx, y\rangle$ is purely imaginary and let $a = e^{i\theta}x$.

Choose n such that $(-1)^n \boldsymbol{Re}\, \langle (A^*A - \delta^2 I)a, y\rangle$ is positive and let $b = (-1)^n y$. Then a and b may be joined by a path in $S(A, \delta)$ to x and y respectively. Thus we need only find a path connecting a to b. Let $y(t) = ta + (1-t)b$ and let $x(t) = ||y(t)||^{-1}y(t)$. Then $\langle Sx(t), x(t)\rangle = 0 \Leftrightarrow \langle Sy(t), y(t)\rangle = 0$ and

$$\begin{aligned}\langle Sy(t), y(t)\rangle &= t^2\langle Sa, a\rangle + (1-t)^2\langle Sb, b\rangle \\ &\quad + 2\boldsymbol{Re}t(1-t)\langle Sa, b\rangle \\ &= 2(-1)^n t(1-t)\boldsymbol{Re}e^{i\theta}\langle Sx, y\rangle \\ &= 0 .\end{aligned}$$

Also

$$\begin{aligned}||Ay(t)||^2 &= \langle A^*Ay(t), y(t)\rangle \\ &= t^2 ||Aa||^2 + (1-t)^2 ||Ab||^2 \\ &\quad + 2t(1-t)\boldsymbol{Re}\langle A^*Aa, b\rangle\end{aligned}$$