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COMPLETIONS OF REGULAR RINGS II

K. R. GOODEARL

This paper continues earlier investigations into the structure of completions of a (von Neumann) regular ring R with respect to pseudo-rank functions, and into the connections between the ring-theoretic structure of such completions and the geometric structure of the compact convex set $P(R)$ of all pseudo-rank functions on R. In particular, earlier results on the completion of R with respect to a single $N \in P(R)$ are extended to completions with respect to any nonempty subset $X \subseteq P(R)$. Completions in this generality are proved to be regular and self-injective by reducing to the case of a single pseudo-rank function, using a theorem that the lattice of σ-convex faces of $P(R)$ forms a complete Boolean algebra. Given a completion $\bar{R}$ with respect to some $X \subseteq P(R)$, it is shown that the Boolean algebra of central idempotents of $\bar{R}$ is naturally isomorphic to the lattice of those σ-convex faces of $P(R)$ which are contained in the σ-convex face generated by X. Consequently, conditions on X are obtained which tell when $\bar{R}$ is a direct product of simple rings, and how many simple ring direct factors $\bar{R}$ must have. Also, it is shown that the X-completion of R contains a natural copy of the completion with respect to any subset of X, so in particular the $P(R)$-completion of R contains copies of all the X-completions of R. The final section investigates the question of when a regular self-injective ring is complete with respect to some family of pseudo-rank functions. It is proved that a regular, right and left self-injective ring R is complete with respect to a family $X \subseteq P(R)$ provided only that the Boolean algebra of central idempotents of R is complete with respect to X.

1. Completions. All rings in this paper are associative with unit, and ring maps are assumed to preserve the unit. This paper is a direct continuation of [7], and the reader should consult [7] for definitions which are not discussed here. A family of pseudo-rank functions on a regular ring R induces a uniform topology on R, and the purpose of this paper is to study the resulting completion of R. We begin by recalling the appropriate topological concepts.

Let S be a nonempty set, and let D be a nonempty family of pseudo-metrics on S. The (uniform) *topology induced by D on S* has as a subbasis the balls $\{x \in S \mid d(x, y) < \varepsilon\}$, for various $y \in S$, $d \in D$, $\varepsilon > 0$. Thus the basic open neighborhoods of a point $y \in S$ are the sets $\{x \in S \mid d_i(x, y) < \varepsilon$ for $i = 1, \cdots, n\}$ for various $\varepsilon > 0$ and $d_1, \cdots, d_n \in D$. A net in S is a *Cauchy net* (with respect to D)