PACIFIC JOURNAL OF MATHEMATICS
Vol. 73, No. 1, 1977

THE CONVERGENCE-PRESERVING REARRANGEMENTS OF REAL INFINITE SERIES

GERALD S. STOLLER

Let $\mathscr{C}$ be the set of all permutations of the natural numbers that carry convergent real infinite series into convergent real infinite series. A strictly algebraic necessary and sufficient condition which determines $\mathscr{C}$ is given. $\mathscr{C}$ is seen to be a monoid but not a group. The maximum subgroup of $\mathscr{C}$ is shown not to be normal in $\mathscr{C}$.

A related set of permutations are those that preserve the sum of a convergent real infinite series when they carry that series to a convergent real series. This set of permutations is not a monoid. By exhibiting three different sufficient conditions for a permutation to belong to this set, we see that necessary and sufficient conditions determining this set will be difficult to ascertain.

Any automorphism of N, the set of nonnegative integers, determines a unique rearrangement of infinite series. This correspondence is an anti-isomorphism. A subset of S_N, the group of all permutations of N, is defined by the condition that it carries convergent real infinite series into convergent real series. An algebraic condition can be shown to specify the same subset. We will see that this subset is a submonoid of S_N, but not a subgroup.

The problem of determining $\mathscr{C}$, the set of all permutations of N that carry convergent real infinite series into convergent real infinite series, was considered nearly impossible to solve by E. Borel [2, page 101]. A necessary and sufficient condition on permutations of N for them to preserve convergence of real infinite series appears in [3, Prob. 5.2.2]. Letting σ denote a permutation of N, this condition requires that there exists a finite cardinal k such that $\sigma([0, n])$ can be expressed as the union of not more than k intervals of N for each $n \in N$. In fact, the set of permutations defined by this condition preserves convergence of infinite series (and their individual sums) in any normed abelian semigroup. Moreover, this is the largest set of permutations that preserve convergence in the family of all normed abelian semigroups. (For some normed abelian groups [e.g., the p-adics] the set of permutations preserving convergence is larger.)

$\mathscr{C}$ is easily shown to be a submonoid by using the convergence preservation property. We show that $\mathscr{C}$ is not a subgroup by establishing the necessary and sufficient conditions of [3, Prob. 5.2.2] and then exhibiting the permutation ρ defined by