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ON THE COMPACTNESS OF THE HYPERSPACE OF FACES

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Let $X \subset R^d$, $d > 2$, be compact and convex. It is shown that the space of proper faces (poonems) of X is compact if and only if the space of k-exposed (extreme) points of X is compact, $0 \leqq k \leqq d-2$.

By a *flat* we mean a translate of a subspace of R^d and by a *hyperplane*, a flat of dimension $d-1$. If X is a compact convex subset of R^d the symbols $\dim(X)$, $\operatorname{rel\,int}(X)$, and $\operatorname{rel\,bd}(X)$ denote respectively, the dimension of the flat generated by X, the interior, and the boundary of X with respect to the flat X generates. A hyperplane H is called a *supporting hyperplane* of X if $H \cap X \neq \emptyset$ and $H \cap \operatorname{relint}(X) = \emptyset$. A set A is called a *face* of X if $A = X$, $A = \emptyset$ or if there exists a supporting hyperplane H of X such that $A = H \cap X$. The set of proper faces (those not X or $\emptyset$) is denoted by $\mathscr{F}(X)$. A set B is called a *poonem* of X if there exists sets $B_0, B_1, \cdots, B_m$ such that $B_m = X$ and $B_{i-1} \in \mathscr{F}(B_i)$ for $i = 1, \cdots, m$. The set of poonems of X is denoted by $\mathscr{P}(X)$. A point x in X is called a k-exposed [k-extreme] point if for some $j \leqq k$, x belongs to a j-dimensional face [j-dimensional poonem] of X. The symbols $\exp_k(X)$ and $\operatorname{ext}_k(X)$ denote the set of k-exposed and k-extreme points of X respectively. A point x of X is called an *exposed point* of X if $\{x\} \in \mathscr{F}(X)$ and x is called an *extreme point* if whenever $x \in [a, b] \subset X$, we have $x = a$ or $x = b$, where $[a, b]$ denotes the closed line segment from a to b. The symbols $\operatorname{ext}(X)$ and $\exp(X)$ denote the extreme points and exposed points of X, respectively. Note that $\exp(X) = \exp_0(X)$ and $\operatorname{ext}(X) = \operatorname{ext}_0(X)$. Also, let q denote the Hausdorff metric. Finally, if $D \subset R^d$, the symbols $\operatorname{cl}(D)$ and $\operatorname{conv}(D)$ denote the closure of D and the convex hull of D respectively.

We require the following results

PROPOSITION. *Let X, A and B be nonempty compact convex subsets of R^d, $d \geqq 2$, with $A \subset X$ and $B \subset X$.*

(a) *If $A, B \in F(X)$ $(P(X))$ and $A \cap B \neq \emptyset$, then $A \cap B \in F(X)$ $(P(X))$.*

(b) *If $A \in F(X)$ $(P(X))$ with relint $(B) \cap A \neq \emptyset$ then $B \subset A$.*

(c) *If $A \in F(X)$ $(P(X))$ and $A \subsetneqq B \subset X$ then $A \in F(B)$ $(P(B))$.*

(d) $\operatorname{ext}_k(X) \subset \operatorname{cl}(\exp_k(X))$.

(e) *If $A \in F(X)$ then* $\operatorname{ext}_k(A) \subset \operatorname{ext}_k(X)$.