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NOTES ON IDEAL COVERS AND ASSOCIATED PRIMES

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The covering relationship between ideals ($B \subset C$ and there exist no ideals between B and C) is studied, and, surprisingly, due to an interesting interplay between the concepts, quite a few new results concerning Bourbaki associated primes are found. Then, most of the results are generalized to submodules of an an arbitrary A-module.

The covering relationship between ideals has proved to be a useful and important tool in many investigations in commutative algebra. For example, a result in a classical paper of Gröbner [2, §6] says that an M-primary ideal Q in a local ring (A, M) is irreducible if and only if Q has a unique cover. (See also [9, p. 248].) (Other examples include most papers where one of the following concepts is considered: the length of an ideal, a minimal basis of an ideal, or the Hilbert-Samuel polynomial of an ideal.) However, the authors know of no paper where the subject itself has been studied. We do such a study in §2 of this paper, and some quite interesting results are obtained. Also, it turns out that there is an interesting interplay between the covering relationship and Bourbaki associated primes, so results concerning one of the concepts imply corresponding results concerning the other concept. Therefore in §3, using the results of §2, quite a few new results concerning Bourbaki associated primes of an ideal in an arbitrary ring are obtained. (In particular, all the results hold for prime divisors of an ideal in a Noetherian ring, and many are new even in this case.) That many new results are obtained is somewhat surprising, since the subject of Bourbaki associated primes has certainly previously been deeply investigated.

To briefly describe the results in §2, let $B \subset C$ be ideals in a ring A. Then we say that C covers B in case there are no ideals D of A such that $B \subset D \subset C$. In this case, $C/B \cong A/M$, for some maximal ideal M, and to emphasize the role of M we say that C M-covers B. With this terminology, it is first shown in §2 that an ideal I in A M-covers some ideal if and only if $MI \neq I$ (2.3), so if A is either quasi-local or an integral domain, then every nonzero finitely generated ideal M-covers some ideal (2.4). Some other corollaries of (2.3) are given in (2.5)-(2.10), among which is a description of an ideal which M-covers a unique ideal — a result which is sort of dual to the above mentioned result of Gröbner. Next it is