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TANGENT WINDING NUMBERS AND BRANCHED MAPPINGS

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The notion of tangent winding number of a regular closed curve on a compact 2-manifold M is investigated, and related to the notion of obstruction to regular homotopy. The approach is via oriented intersection theory. For N, a 2-manifold with boundary and $F: N \to M$ a smooth branched mapping, a theorem is proved relating the total branch point multiplicity of F and the tangent winding number of $F|_{\partial N}$. The theorem is a generalization of the classical Riemann-Hurwitz theorem.

1. Introduction. Let M be a smooth, connected, oriented 2-manifold and let f and g be regular closed curves on M with the same initial point and tangent direction. An integer obstruction to regular homotopy $\gamma(f, g)$ is derived which is uniquely defined if $M \neq S^2$ and defined mod 2 if $M = S^2$. Let $F(t, \theta)$ be any homotopy such that $F(0, \theta) = f(\theta)$ and $F(1, \theta) = g(\theta)$ and F is smooth on the interior of the unit square. It is shown that $\gamma(f, g) = I(\partial F/\partial\theta, M_0)$, where M_0 is the zero section as a sub-manifold of TM, and I denotes the total number of oriented intersections. This is given interpretation as the number of loops acquired by curves $F(t, \,) = f_t$ in homotopy.

If M is compact and y is not on the image of f, then we define $\operatorname{twn}(f; y)$, a generalization of the tangent winding number. We show that $\gamma(f, g) = \operatorname{twn}(g; y) - \operatorname{twn}(f; y) + I(F, y)\chi(M)$, where is the Euler characteristic. If N is a 2-manifold with boundary and $F: N \to M$ is a smooth branched mapping and $\partial F = F|_{\partial N}$, we show that $\operatorname{twn}(\partial F; y) + I(F, y)\chi(M) = \chi(N) + r$, where r is the total branchpoint multiplicity and y is not in $F(\partial N)$. We show that the Riemann-Hurwitz theorem follows as a corrollary.

2. The obstruction to regular homotopy. Let M be a smooth, connected 2-manifold with Riemannian metric. Let TM be the tangent bundle and $\tilde{T}M$ the unit tangent or sphere bundle. Let $f: R \to M$ with $f(\theta) = f(\theta + 1)$ for all $\theta \in R$ be a regular closed curve on M, that is, f has continuously turning, nonzero tangent vector at each point. Given $F: [0, 1] \times R \to M$ continuous with $F(t, \theta) = F(t, \theta + 1)$ for all $\theta \in R$, then F is said to be a regular homotopy if each closed curve $F(t, \,)$ is regular for $0 \leqq t \leqq 1$. We say the curves $f(\theta) = F(0, \theta)$ and $g(\theta) = F(1, \theta)$ are regularly homo-