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LIE ALGEBRAS WITH DESCENDING CHAIN CONDITION

J. MARSHALL OSBORN

In this note we investigate Lie algebras which satisfy the descending chain condition on ideals of ideals. We show that a Lie algebra L satisfies this descending chain condition if and only if the following two conditions hold: (i) L contains a finite dimensional solvable ideal N such that every solvable ideal of L is contained in N, and (ii) L/N is a subdirect sum of a finite number of prime algebras satisfying the descending chain condition. We also show that if L is a prime algebra with this chain condition then there exists a Lie algebra B, which is either simple or the tensor product of a simple Lie algebra with a truncated polynomial algebra, such that L is isomorphic to a subalgebra of Der B containing ad_B.

A decade ago a theory of Jordan algebras with descending chain condition on inner ideals was developed [3, Chapter IV] which emulates and connects with the theory of Artinian rings. More recently Benkart [1] studied Lie algebras with descending chain condition on inner ideals (a subspace B of a Lie algebra L is called an *inner ideal* of L if $[B, [B, L]] \subseteq B$). It has not been settled yet whether a Jordan algebra with DCC on inner ideals necessarily has a nilpotent radical. One of the purposes of the present paper is to show that a Lie algebra with DCC on inner ideals has a radical which is solvable and finite dimensional. This follows from the results stated in the last paragraph since any ideal of an ideal is an inner ideal and hence DCC on inner ideals implies DCC on ideals of ideals.

It is known that a finite dimensional semisimple Lie algebra M of characteristic p is not necessarily a direct sum of simple algebras, but there do not seem to be any results published which express M in terms of algebras which belong to a more restricted class than M. A second purpose of this paper is to show that M is a subdirect sum of prime algebras. Rather than finite dimensionality the assumption of DCC on ideals of ideals seems to be the most natural level of generality for this proof.

The results in this paper hold for Lie algebras over a field Φ of any characteristic including 2.

Suppose now that L is a Lie algebra with DCC on ideals of ideals. We begin with

LEMMA 1. *If C is a solvable ideal of L, then C is finite*