PACIFIC JOURNAL OF MATHEMATICS
Vol. 73, No. 1, 1977

MERCERIAN THEOREMS VIA SPECTRAL THEORY

FRANK P. CASS AND B. E. RHOADES

Given a regular matrix A, Mercerian theorems are concerned with determining the real or complex values of α for which $\alpha I + (1-\alpha)A$ is equivalent to convergence. For $\alpha \neq 1$, the problem is equivalent to determining the resolvent set for A, or, determining the spectrum $\sigma(A)$ of A, where $\sigma(A) = \{\lambda \mid A - \lambda I$ is not invertible$\}$. This paper treats the problem of determining the spectra of weighted mean methods; i.e., triangular matrices $A = (a_{nk})$ with $a_{nk} = p_k/P_n$, where $p_0 > 0$, $p_n \geqq 0$, $\sum_{k=0}^{n} p_k = P_n$. It is shown that the spectrum of every weighted mean method is contained in the disc $\{\lambda \mid |\lambda - 1/2| \leqq 1/2\}$ (Theorem 1), and, if $\lim p_n/P_n$ exists,

$$\sigma(A) = \{\lambda \mid |\lambda - (2-\varepsilon)^{-1}| \leqq (1-\varepsilon)/(2-\varepsilon)\} \cup \{p_n/P_n \mid p_n/P_n < \varepsilon/(2-\varepsilon)\} ,$$

where $\varepsilon = \lim p_n/P_n$.

Let $\gamma = \underline{\lim}\, p_n/P_n$, $\delta = \overline{\lim}\, p_n/P_n$, $S = \overline{\{p_n/P_n \mid n \geqq 0\}}$. When $\gamma < \delta$, some examples are provided to indicate the difficulty of determining the spectrum explicitly. It is shown that $\{\lambda \mid |\lambda - (2-\delta)^{-1}| \leqq (1-\delta/(2-\delta))\} \cup S \subseteqq \sigma(A)$ and

$$\sigma(A) \subseteqq \{\lambda \mid |\lambda - (2-\gamma)^{-1}| \leqq (1-\gamma)/(2-\gamma)\} \cup S .$$

Theorem 1 is a generalization of the corresponding theorems of: S. Aljancic, L. N. Cakalov, K. Knopp, M. E. Landau, J. Mercer, Y. Okada, W. Sierpinski, and G. Sunouchi.

Using spectral theory we obtain the best possible Mercerian theorems for certain classes of weighted mean methods of summability.

The weighted mean method is a triangular matrix $A = (a_{nk})$ with $a_{nk} = p_k/P_n$, where $p_0 > 0$, $p_n \geqq 0$, $n \geqq 0$, $P_n = \sum_{k=0}^{n} p_k$ and A is a bounded linear operator on c, the space of convergent sequences.

For $\alpha \neq 0$ we may write $\alpha I + (1-\alpha)A = \alpha(I + qA)$, where $q = (1-\alpha)/\alpha$. Mercer's original theorem [9] states the following: Let $\{x_n\}$ be a sequence such that $x_{n+1} - x_n + \mu n^{-1} x_n \to \lambda$ as $n \to \infty$. (i) If λ is finite and $\mu > -1$, then $x_{n+1} - x_n$ and $n^{-1}x_n$ both tend to $\lambda/(\mu+1)$ as $n \to \infty$. (ii) If λ is infinite and $\mu > -1$, then $n^{-1}x_n \to \lambda$ and $x_{n+1} - x_n \to \lambda$ only if $0 \geqq \mu > -1$.

Landau [8] showed that, if $\{x_n\}$ is a complex sequence, q a positive integer, then $\lim_n (x_n + (q/n) \sum_{k=1}^{n} x_k) = 0$ implies $\lim_n x_n = 0$. Sierpinski [14] extended Landau's result to real numbers $q > -1$ and showed it could not be extended to $q \leqq -1$. Sierpinski's result for $q > -1$ was reproved in [3].