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ON A CONNECTION BETWEEN NILPOTENT GROUPS AND OSCILLATORY INTEGRALS ASSOCIATED TO SINGULARITIES

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The object of this paper is to demonstrate and promote some ties between the theory of harmonic analysis on nilpotent Lie groups theory and another topic the study of oscillatory integrals associated to polynomial singularities. Oscillatory integrals are tempered distributions on $\boldsymbol{R}^n$, defined by integration against the exponentials of (real-valued) polynomials.

Thus if p is polynomial on $\boldsymbol{R}^n$ with real coefficients, the associated oscillatory integral is

$$E_p(f) = \int_{\boldsymbol{R}^n} f(x)e^{ip(x)}dx \tag{1}$$

where f belongs to $\mathscr{S}(\boldsymbol{R}^n)$, the Schwartz space of $\boldsymbol{R}^n$, and dx is Lebesgue measure. The main questions of which I am aware concerning the distributions E_p are two.

(a) Asymptotic behavior: For $t \in \boldsymbol{R}$, how does $E_{tp}(f)$ behave as $t \longrightarrow \infty$? In particular, what is the slowest rate of decay of $E_{tp}(f)$?

(b) Fourier transform: Define the Fourier transform $\hat{}: \mathscr{S}(\boldsymbol{R}^n) \longrightarrow \mathscr{S}(\boldsymbol{R}^n)$ by the usual formula

$$\hat{f}(x) = \int_{\boldsymbol{R}^n} f(y)e^{-2\pi i x\cdot y}dy \tag{2}$$

where $x\cdot y$ is the usual inner product on $\boldsymbol{R}^n$. Define $\hat{E}_p$ by

$$\hat{E}_p(f) = \int_{\boldsymbol{R}^n} \hat{f}(-x)e^{ip(x)}dx . \tag{3}$$

Can $\hat{E}_p$ be represented directly as a distribution? Is it given as integration against some function? How can this function be described if it exists?

The simplest case of interest for questions (a) and (b) is that of the stable or Morse singularities, when p is a nondegenerate quadratic form. Here both questions (a) and (b) have well-known, satisfying, classical answers [5]. We recall the formulas in one-dimension.

$$(e^{\pi i t x^2})\hat{} = \left(\frac{1+i}{\sqrt{2t}}\right)e^{-\pi i x^2/t}\ , \quad t > 0 \tag{4}$$