PACIFIC JOURNAL OF MATHEMATICS
Vol. 73, No. 2, 1977

THE FOURIER TRANSFORM FOR NILPOTENT LOCALLY COMPACT GROUPS: I

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In his work on nilpotent lie groups, A. A. Kirillov introduced the idea of classifying the representations of such groups by matching them with orbits in the dual of the lie algebra under the coadjoint action. His methods have proved extremely fruitful, and subsequent authors have refined and extended them to the point where they provide highly satisfactory explanations of many aspects of the harmonic analysis of various lie groups. Meanwhile, it appears that nonlie groups are also amenable to such an approach. In this paper, we seek to indicate that, indeed, a very large class of separable, locally compact, nilpotent groups have a Kirillov-type theory.

On the other hand, elementary examples show that not all such groups can have a perfect Kirillov theory. The precise boundary between good and bad groups is not well defined, and varies with the amount of technical complication you can tolerate. At this stage, the delineation of the boundary is the less rewarding part of the theory, and will be deferred to a future publication. In the present paper, we lay some groundwork, and then discuss a particularly nice special case, which also has significance in the general picture.

Since Kirillov's approach hinges on the use of the lie algebra and its dual, the first concern in imitating his theory is to find a lie algebra. In §I, we consider the generalities of the algebraic aspects of this problem. We rely very heavily on Serre [10]. Indeed, the beginning of §I is a summary of the fourth chapter of [10], with differences in emphasis to fit the present need. The major tool is the Campbell-Hausdorff formula, which we use to prove some elementary facts on nilpotent groups, as well as to construct lie algebras.

In §II, we discuss the structure theory of locally compact nilpotent groups. We should emphasize here that we are dealing with groups which are genuinely nilpotent in the algebraic sense, and which have a topology. We do not consider groups which are nilpotent only in some topological sense. Specifically, G is k-step nilpotent if the ascending central series $\mathscr{Z}(G)$, $\mathscr{Z}^{(2)}(G)\cdots$, satisfies $\mathscr{Z}^{(k)}(G) = G$. Alternatively, if $x, y \in G$, define the commutator (x, y) of x and y by $(x, y) = x^{-1}y^{-1}xy$. Define the order of a commutator inductively: all $x \in G$ have order one; the commutator of commutators of orders i