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ON REPRESENTATIONS OF DISCRETE, FINITELY GENERATED, TORSION-FREE, NILPOTENT GROUPS

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With A. A. Kirillov's work on the representations of nilpotent lie groups, a new chapter in the theory of group representations opened. Subsequent papers of Bernat, Moore and Auslander-Kostant have further demonstrated the power of the methods introduced by Kirillov. The purpose of this paper is to begin an extension of these methods in yet another direction. Specifically, the object here is to calculate the primitive ideal spaces of the groups indicated in the title.

While this again is a fairly special extension of what eventually should be a very far-reaching theory, it has present merit for several reasons: (1) Much work necessary for further extensions is already done in handling these particular groups. In particular, slight modifications and extensions of these methods allow one to deal with most unipotent groups that occur in arithmetic; (2) Most of these groups are not type I (see Thoma [14]); hence they provide examples of what can be said about such "bad" groups; (3) The theory of these groups also sheds light on the harmonic analysis of finite p-groups.

From now on, Γ will always denote a discrete, finitely generated, torsion-free, nilpotent group. e will be its identity element, $\Gamma = \Gamma^{(1)}$, $\Gamma^{(2)}$, $\Gamma^{(3)}$, etc. its descending central series, and $\mathscr{Z}(\Gamma)$, $\mathscr{Z}^{(2)}(\Gamma)$, etc. its ascending central series. If $\Gamma^{(k+1)} = \{e\}$, we will say Γ is k-step nilpotent.

To carry out for Γ a program analogous to Kirillov's we need for Γ a dual of a lie algebra, and so, first, a lie algebra. Malcev (see [1]) has shown that any Γ such as we are considering may be embedded as a discrete subgroup in a simply connected nilpotent lie group $\mathscr{N}$, so that the quotient $\mathscr{N}/\Gamma$ is compact; furthermore $\mathscr{N}$ and the embedding are unique up to isomorphism. From now on, $\mathscr{N}$ will always denote this group, and we will consider Γ to be a subgroup of $\mathscr{N}$ whenever this is convenient.

N will denote the lie algebra of $\mathscr{N}$. Then $\exp: N \rightarrow \mathscr{N}$ is a diffeomorphism, with log as its inverse. We would like to say $\log \Gamma \subseteq N$ is the lie algebra of Γ. However, this makes no sense in general, because $\log \Gamma$ need be closed neither under addition, nor under taking commutators. But suppose $\mathscr{N}$ is k-step nilpotent, and $L \subseteq N$ is a lattice, such that $[L, L] \subseteq k!L$. Then an easy calculation with the Campbell-Hausdorff formula ([13]) shows that $\Gamma =$