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A REMARK ON FUNCTIONS OF BOUNDED MEAN OSCILLATION AND BOUNDED HARMONIC FUNCTIONS

N. TH. VAROPOULOS

This note is an Addendum to [7]; it is written, however, in such a way that it can be read independently.

We recall that μ a positive measure on the upper half space $\boldsymbol{R}_+^{n+1} = \{(x, y); x \in \boldsymbol{R}^n, y > 0\}$ (the notations we use are those of [6]) is called a Carleson measure if

$$\mu\{x \in I, 0 < y < h\} \leqq Ch^n$$

for all I hypercube in $\boldsymbol{R}^n$ of side length equal to h (cf. [6] VII 44, [1] [7]). In this note we shall prove the following

THEOREM 1. *Let u be a real bounded harmonic function in $\boldsymbol{R}_+^{n+1}$ such that $\|u\|_\infty \leqq 1$ then there exists $v \in C^\infty(\boldsymbol{R}_+^{n+1})$ a real bounded smooth function such that $|\nabla v| d$ Vol is a Carleson measure in $\boldsymbol{R}_+^{n+1}$ and such that $\|u - v\|_\infty \leqq 1 - \varepsilon_0$ for some numerical constant ε_0* $(0 < \varepsilon_0 < 1)$.

∇v denotes of course the gradient of v in $\boldsymbol{R}_+^{n+1}$.

REMARK. Note that $|\nabla u| d$ Vol is not in general a Carleson measure (cf. [5]).

THEOREM 2. *Let $f \in$ BMO$(\boldsymbol{R}^n)$ then there exists $F \in C^\infty(\boldsymbol{R}_+^{n+1})$ such that*

(i) $F(x, y) \xrightarrow[y \to 0]{} f(x)$ $x \in \boldsymbol{R}^n$ *p.p.*

(ii) $\sup_{y>0} |F(x, y)| \in L^1_{\text{loc}}(\boldsymbol{R}^n; dx)$

(iii) $|\nabla F| d$ Vol *is a Carleson measure in* $\boldsymbol{R}_+^{n+1}$.

Conversely if $f \in L^1_{\text{loc}}(\boldsymbol{R}^n)$ is such that there exists some $F \in C^1(\boldsymbol{R}_+^{n+1})$ that satisfies (i), (ii), *and* (iii) *then f is a* BMO *function.*

(For the definition of BMO and background cf. [2].)

In functional terms the above theorem means that the space BMO is the restriction space (i.e., quotient space) of an appropriate space of C^∞ functions in the upper half space (cf. [3] for analogous results).

Both the above theorems can be generalized in other contexts,