

A NOTE ON RADON-NIKODYN THEOREM FOR FINITELY ADDITIVE MEASURES

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**The Radon-Nikodym theorem for finitely additive measures
 is deduced from the corresponding result for countably addi-
 tive measures.**

In ([4], Theorem 1, p. 35) a Radon-Nikodym type result is proved for finitely additive measures. In this note we prove that this result is a simple consequence of the corresponding result for the countably additive case.

Let \mathcal{A}_0 be an algebra of subsets of a set X ; without loss of generality we assume that \mathcal{A}_0 is reduced, i.e., separates points of X ([5], p. 68). We denote by ρ the isomorphism between \mathcal{A}_0 and \mathcal{A} the algebra of all clopen subsets of \hat{X} , the compact Hausdorff, totally disconnected space which is the Boolean space for \mathcal{A}_0 ([5], p. 70).

THEOREM ([4], Theorem 1, p. 35). *Let λ and μ be two complex-valued finite-additive measures on \mathcal{A}_0 such that μ is bounded and λ is absolutely continuous relative to μ ($\varepsilon - \delta$ meaning of absolute continuity). Then there exists a sequence $\{f_n\}$ of \mathcal{A}_0 -simple functions on X such that*

$$(1) \quad \lim \int_A f_n d\mu = \lambda(A), \quad \text{unif. for } A \in \mathcal{A}_0$$

and

$$(2) \quad \lim_{m, n \rightarrow \infty} \int |f_n - f_m| d|\mu| = 0, \quad |\mu| \text{ being the total variation of } \mu \text{ ([2]).}$$

Proof. For any disjoint sequence $\{A_n\} \subset \mathcal{A}_0$, $|\mu|(A_n) \rightarrow 0$ (note μ is bounded) and so $\lambda(A_n) \rightarrow 0$. This means λ is exhaustive (\equiv strongly bounded) and so λ is bounded ([1]). λ and μ naturally give rise to countably additive measures λ' and μ' on \mathcal{A} and as such can be uniquely extended to the σ -algebra \mathcal{B}_∞ generated by \mathcal{A} ; \mathcal{B}_∞ is also the class of all Baire subsets of \hat{X} ([5], p. 70). We claim $|\lambda'|$ is absolutely continuous with respect to $|\mu'|$: suppose $|\mu'| (B) = 0$ but $|\lambda'| (B) > 0$ for some $B \in \mathcal{B}_\infty$. This means there exists a $C \subset B$, $C \in \mathcal{B}_\infty$ such that $|\lambda'(C)| > \varepsilon$ for some $\varepsilon > 0$. Fix $\delta > 0$ such that $P \in \mathcal{A}_0$, $|\mu|(P) < \delta$ implies $|\lambda(P)| < \varepsilon$. Since Baire measures are regular, there exists an open subset V of \hat{X} such that $V \supset C$, $|\mu'(V)| < \delta$, and $|\lambda'(V)| > \varepsilon$. Again by regularity and total disconnectedness of \hat{X} there is a clopen subset $U \subset V$ such that $|\mu'(U)| < \delta$ and $|\lambda'(U)| > \varepsilon$. Taking $P = \rho^{-1}(U)$ we get $|\mu|(P) < \delta$ and $|\lambda(P)| > \varepsilon$, a contradiction.