

ON A THEOREM OF S. BERNSTEIN

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In 1912, S. Bernstein, in the first part of his memoir [2] devoted to the boundary value problems arising in calculus of variations, established sufficient conditions for the unique solvability of the Dirichlet problem for the equation $y'' = f(t, y, y')$. Our aim is to present a result which extends the scope of the Bernstein theorem and to show that the generalization obtained can be carried over (with only minor adjustments in the proof) to the case of all important boundary value problems which arise in applications.

1. Introduction. In this paper we study the existence and uniqueness problems for a second order differential equation of the form

$$(*) \quad y'' = f(t, y, y'), \quad 0 \leq t \leq 1,$$

subject to certain boundary conditions, labeled (I)-(VII) below. These boundary conditions include the Dirichlet, Neumann, and periodic ones as well as the so-called elastic or Sturm-Liouville boundary conditions. We only treat in detail homogeneous boundary conditions; however, the case of inhomogeneous boundary conditions can be treated similarly to the homogeneous case. See 7(d).

The Dirichlet, Neumann, and periodic boundary conditions are, respectively,

$$\begin{aligned} \text{(I)} \quad & y(0) = 0, \quad y(1) = 0; \\ \text{(II)} \quad & y'(0) = 0, \quad y'(1) = 0; \\ \text{(III)} \quad & y(0) = y(1), \quad y'(0) = y'(1). \end{aligned}$$

The problem of solving the differential equation (*) subject to the boundary conditions (I) will be referred to as problem (I). Similar notation is used for the other problems.

We always assume that $f(t, y, p)$ is defined and continuous on $[0, 1] \times R \times R$. By a solution to problem (I), we mean a function $y \in C^2[0, 1]$ which satisfies the differential equation and boundary conditions. Likewise, we seek $C^2[0, 1]$ solutions to the other problems.

In 1912, S. Bernstein established the following theorem in [2] for problem (I):

Assume $f = f(t, y, p)$ is continuous, has continuous partial derivatives f_y and f_p , and satisfies

- (i) $f_y \geq k > 0$
- (ii) $|f(t, y, p)| \leq A(t, y)p^2 + B(t, y),$