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AN INEQUALITY INVOLVING THE LENGTH, CURVATURE, AND TORSIONS OF A CURVE IN EUCLIDEAN n-SPACE

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Let x be a closed nondegenerate C^n curve in E^n parametrized by arc length s. We prove an inequality for such x which lie in a ball of radius R. For nonplanar curves in E^3 the inequality is

$$L \leqq R^2 \frac{\int_0^L \kappa^2 ds \int_0^L \tau^2 ds - \left(\int_0^L \kappa\tau ds\right)^2}{\int_0^L \tau^2 ds}$$

where L is the length of x, and κ and τ are the curvature and torsion of x, respectively. Equality holds only if x is a great circle on a sphere of radius R. We also obtain from the general inequality necessary conditions on the length, curvature, and torsions of x in order that x be a closed curve or a closed curve with at most one corner.

1. Definitions. We say a C^n curve x in E^n is *nondegenerate* if it has a Frenet framing. That is, there exists an orthonormal set of vector fields $e_1, e_2, \cdots, e_n$ along x such that

$$(1)\qquad \begin{aligned} x' &= e_1 \\ e_1' &= \kappa e_2 \\ e_2' &= -\kappa e_1 + \tau_1 e_3 \\ e_3' &= -\tau_1 e_2 + \tau_2 e_4 \\ &\vdots \\ e_n' &= -\tau_{n-2} e_{n-1}\,, \end{aligned}$$

where the prime denotes differentiation with respect to arc length, κ is the curvature, and $\tau_1, \tau_2, \cdots, \tau_{n-2}$ are the torsions of x. For the remainder of this paper, we assume that x is nondegenerate and $\tau_i \neq 0$, for $i = 1, 2, \cdots, n-2$. In what follows we also let $\tau_0 = \kappa$ and $\tau_{n-1} = 0$.

We say $x: [0, L] \to E^n$ is closed if it induces a C^n mapping $x: S^1 \to E^n$, where S^1 is the circle. To say $x: [0, L] \to E^n$ is closed with at most one corner means that $x(0) = x(L)$ but $x'(0)$ need not equal $x'(L)$.

Define $x_i = (x, e_i)$, for $i = 1, 2, \cdots, n$, where (,) denotes the inner product in E^n. Then from (1) we obtain