PACIFIC JOURNAL OF MATHEMATICS
Vol. 74, No. 2, 1978

SIMPLIFYING SPINES OF 3-MANIFOLDS

RICHARD OSBORNE

It is well known that every compact 3-manifold has a spine that is a 2-dimensional cell complex with just one vertex. Such a cell complex determines a group presentation in a natural way. It seems natural to call K a simpler spine than K' if the presentation corresponding to K is shorter than that corresponding to K'. In this paper we give an algebraic condition which is sufficient to guarantee the existence of a simpler spine.

Of course, identifying the simplest spine of a 3-manifold would allow one to solve the homeomorphism problem for 3-manifolds. From one point of view the difficulty with identifying the simplest spines arises from the lack of correspondence between algebraic operations on presentations and geometric alteration of spines. For example $\langle a, b \mid a^2b^3, a^3b^4 \rangle$ corresponds to a spine of S^3 but $\langle a, b \mid a^2b^3a^3b^4, a^3b^4 \rangle$ does *not* correspond to the spine of any 3-manifold. (See [7] for verification of this fact.) To state our result we need some definitions. Let $\phi = \langle X \mid \mathscr{R} \rangle$ be a group presentation, $X = \{x_1, x_2, \cdots, x_n\}$ and $\mathscr{R} = \{R_1, R_2, \cdots, R_k\}$ $\mathscr{R}$ being a set of words in the free semigroup on $X \cup X^{-1}$. In what follows we will not distinguish between a relator R_i and any cyclic conjugate of it or its inverse. This convention is adopted because the complexes determined are the same. Let $\phi' = \langle X \mid R_1, R_2, \cdots, R_{i-1}, R_iR_j, R_{i+1}, \cdots, R_k \rangle$ where $j \neq i$. We shall say that ϕ' was obtained from ϕ by *multiplication* of R_i and R_j. If $\mathscr{A}$ is an automorphism of $F(X)$ (the free group on X) we denote by $\mathscr{A}(R_i)$ the image of R_i under $\mathscr{A}$. We denote by $\mathscr{A}(\phi)$ the presentation $\langle X \mid \mathscr{A}(R_1), \cdots, \mathscr{A}(R_k) \rangle$. The *length* of a presentation ϕ is the sum of the lengths of the freely reduced relators of ϕ.

THEOREM 1. *If K_ϕ is the spine of a 3-manifold M with corresponding presentation ϕ, $\bar{\phi}$ is obtained from ϕ by automorphism or multiplication and the length of $\bar{\phi}$ is less than the length of ϕ then M has a simpler spine than K_ϕ. This spine has a presentation ϕ' that can be obtained from ϕ by automorphism and or by multiplication. One can assume that ϕ' is at least as short as $\bar{\phi}$ if $\bar{\phi}$ was obtained from ϕ by automorphism or if ϕ could not be reduced in length by an automorphism.*

A proof of this theorem appears in §3. Actually a somewhat stronger theorem is proved, as will be pointed out. We now give some examples to illustrate Theorem 1.