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BOUNDED MONOIDS

WILLIAM R. NICO

Monoids whose left S-sets X always satisfy $sl(X) \leqq h(X)$ are characterized in terms of chain conditions on principal left ideals.

For S a monoid, a left S-set (S-operand, S-system) is a set X on which S operates from the left and such that $1x = x$ for all $x \in X$ where $1 \in S$ is the identity. For any $x \in X$, an S-subset of X of the form Sx is called an orbit of X. It is well-known that every left S-set is a union of orbits and that, up to isomorphism, orbits are characterized by left congruences on S (see [1], Chapt. 11).

In order to study the way orbits fit together in an S-set X the author has in [2] and [3] constructed two chains of S-subsets (to be defined more fully below) of an S-set X, each having the property that the subquotients are essentially 0-disjoint unions of orbits. The lengths of those two chains are denoted by $h(X)$ and $sl(X)$. In [3] it is shown that when $h(X)$ is finite, then $sl(X) \leqq h(X)$.

Let us call a monoid *bounded* if for every S-set X, one has $sl(X) \leqq h(X)$. Then a main goal of this paper is to show that a monoid is bounded if and only if there is a positive integer n such that the monoid contains no proper chain of principal left ideals of length exceeding n.

1. Preliminaries. Let X be a left S-set. An S-subset Y of X is a (possibly empty) subset Y of X such that $sy \in Y$ for all $s \in S$ and $y \in Y$. If X and Y are both orbits, we may say that Y is a suborbit of X. If Z is an S-set, a homomorphism $\phi: X \rightarrow Z$ is a function such that $\phi(sx) = s\phi(x)$ for all $x \in X$ and $s \in S$. A congruence $\sim$ on an S-set X is an equivalence relation such that $x \sim y$ implies $sx \sim sy$ for $x, y \in X$ and $s \in S$. Denoting the set of congruence classes by $X/\sim$ one finds that $X/\sim$ is a left S-set under the induced action and is a homomorphic image of X under the natural map $X \rightarrow X/\sim$.

If $Y \subset X$ is an S-subset, define a congruence $\sim_Y$ on X by setting $x \sim_Y x'$ if and only if $x = x'$ or $x, x' \in Y$. Let us denote $X/\sim_Y$ simply by X/Y. If $Y \neq \emptyset$, the class of Y in X/Y is denoted by 0.

If X is a left S-set, call $x, y \in X$ *separated in* X if there is no $z \in X$ such that $x, y \in Sz$. Then let us define a descending chain of S-subsets of X by setting $X_0 = X$, $X_{i+1} = \cup\{Sx \cap Sy: x, y \in X_i$ are separated in $X_i\}$ for $i > 0$, and $X_\sigma = \cap\{X_\tau: \tau < \sigma\}$ for σ a limit ordinal. Then $sl(X)$, the *saturation length* of X, is the first ordinal