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EQUATIONAL DEFINABILITY OF ADDITION IN CERTAIN RINGS

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Boolean rings and Boolean algebras, though historically and conceptually different, were shown by Stone to be *equationally* interdefinable. Indeed, in a Boolean ring, addition can be defined in terms of the ring multiplication and the successor operation (Boolean complementation) $x^\wedge = 1 + x(=1 - x)$. In this paper, it is shown that this type of equational definability of addition also holds in a much wider class of rings, namely periodic rings (ring satisfying $x^m = x^n$, $m \neq n$) in which the idempotent elements are "well behaved." More generally, the following theorem is proved:

Suppose R is a ring with unity 1, not necessarily commutative. Suppose further that R satisfies the identity $x^n = x^{n+1}f(x)$ where n is a fixed positive integer and $f(x)$ is a fixed polynomial with integer coefficients. If, further, the idempotent elements of R commute with each other, then addition in R is equationally definable in terms of multiplication in R and the successor operation $x^\wedge = 1 + x$.

Some new classes of rings to which this theorem applies are exhibited.

1. The periodic case. In this section, we shall consider a periodic ring R with unity 1 in which the idempotent elements commute with each other, and will give a *direct* proof of the equational definability of the "+" of R in terms of "$\times$" and the successor operation $x^\wedge$. This direct proof avoids the axiom of choice. We begin with a formal definition of a *periodic* ring.

DEFINITION 1. A ring R is called *periodic* if there exist fixed integers m and n with $m > n \geqq 1$ such that for all x in R, $x^n = x^m$.

LEMMA 1. *Let R be a periodic ring with unity* 1. *Then* (i) *For each x in R, $x^{(m-n)n}$ is idempotent.* (ii) *x is nilpotent if, and only if $x^n = 0$.*

Proof. (i) It can be shown by induction that the identity $x^n = x^m$ $(m > n \geqq 1)$ implies that for all positive integers r

$$x^n = x^{n+r}(x^{m-n-1})^r \,. \tag{1}$$

In particular $x^n = x^{2n}(x^{m-n-1})^n$. Let $e = (x^{m-n})^n$. It is readily verified that $e^2 = e$, which proves (i). Part (ii) follows at once from equation (1).