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EXTREMAL PROPERTIES OF REAL BIAXIALLY SYMMETRIC POTENTIALS IN $E^{2(\alpha+\beta+2)}$

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The set $\mathscr{B}$ consists of all real biaxially symmetric potentials $U^{(\alpha,\beta)}(x, y) = \sum_{n=0}^{\infty} a_n(x^2+y^2)^n P_n^{(\alpha,\beta)}(x^2-y^2/x^2+y^2)/P_n^{(\alpha,\beta)}(1)$, $\alpha > \beta > -1/2$ which are regular in the open unit sphere Σ about the origin in $E^{2(\alpha+\beta+2)}$. Three problems appear regarding $\mathscr{B}$ and subset $\mathscr{B}_*$ whose members have the first $m+1$ coefficients $a_0, \cdots, a_m$ specified. (1) For $U^{(\alpha,\beta)} \in \mathscr{B}$, determine $I(U^{(\alpha,\beta)}) = \inf\{U^{(\alpha,\beta)}(x, y) \mid (x, y) \in \Sigma\}$ as limit of a monotone sequence of constants $\{\lambda_{2n}(a_0, \cdots, a_n)\}_{n=0}^{\infty}$ which can be computed algebraically. (2) Find $U_0^{(\alpha,\beta)} \in \mathscr{B}_*$ and the constant $\lambda_{2m}(a_0, \cdots, a_m) = \sup\{I(U^{(\alpha,\beta)}) \mid U^{(\alpha,\beta)} \in \mathscr{B}_*\} = I(U_0^{(\alpha,\beta)})$. (3) Determine necessary and sufficient conditions from the Fourier coefficients so that $U^{(\alpha,\beta)} \in \mathscr{B}$ and $U^{(\alpha,\beta)}$ is nonnegative in Σ. We develop solutions using operators based on Koornwinder's Laplace type integral for Jacobi polynomials, along with applications of the methods of ascent and descent to the Caratheodory-Fejer and Caratheodory-Toeplitz problems which focus on the properties of harmonic functions in E^2.

1. Introduction. Real biaxially symmetric potentials (BASP) $U^{(\alpha,\beta)}$ which are regular in some domain Ω about the origin in $E^{2(\alpha+\beta+2)}$ may be expanded uniquely as a series

$$U^{(\alpha,\beta)}(x, y) = a_0 + 2\sum_{n=1}^{\infty} a_n U_n^{(\alpha,\beta)}(x, y) , \quad \alpha, \beta > -1/2 \tag{1}$$

in terms of the complete set of biaxially symmetric harmonic polynomials

$$U_n^{(\alpha,\beta)}(x, y) = (x^2 + y^2)^n P_n^{(\alpha,\beta)}(x^2 - y^2/x^2 + y^2)/P_n^{(\alpha,\beta)}(1) , \tag{2}$$

defined from the Jacobi polynomials [1, p. 9]. These functions are necessarily even, satisfying the Cauchy data

$$U_x^{(\alpha,\beta)}(0, y) = U_y^{(\alpha,\beta)}(x, 0) = 0 \tag{3}$$

along the singular lines $x = 0$, $y = 0$ in Ω.

Symmetry about one axis reduces $U_n^{(\alpha,\beta)}$ to zonal harmonics ($\alpha=\beta$), identifying $U^{(\alpha,\beta)}$ as a generalized axially symmetric potential (GASP) [1, p. 10; 5, p. 167] which corresponds to the real part of an analytic function of one complex variable when $\alpha = \beta = -1/2$. This simple correspondence provides characterizations of the fundamental properties of harmonic functions in E^2 from their Fourier coefficients in circular harmonics as they are determined by those of the as-