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A SOLUTION FOR SCATTERED ORDER TYPES OF A PROBLEM OF HAGENDORF

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J. Hagendorf asked if every order type ϕ having the following two properties of additively indecomposable ordinals was the order type of an ordinal. Call ϕ Hagendorf if (i) it is strictly indecomposable to the right, i.e., if $\phi = \psi + \theta$, then ϕ can be embedded in θ but not in ψ, and (ii) every strictly smaller type can be embedded in an initial segment of ϕ, i.e., if χ can be embedded in ϕ but not vice versa, then $\phi = \psi + \theta$ where $\theta \neq 0$ and χ can be embedded in ψ. Recall that scattered order types are those which do not embed the order type of the rationals.

The paper provides a partial answer to Hagendorf's question: *Every scattered Hagendorf type is the order type of an indecomposable ordinal.*

Other subclasses of order types for which this question seems particularly interesting are sub-types of the order type of the real numbers, and the class of countable unions of scattered types.

If ϕ and ψ are order types of linearly ordered sets, then say ϕ is embeddable in ψ, and write $\phi \leqq \psi$ if for any representatives L and M of ϕ and ψ respectively, there is an embedding (one-to-one order-preserving function) of L into M. Write $\phi < \psi$ if $\phi \leqq \psi$ and $\psi \nleqq \phi$.

Above strictly indecomposable to the right is defined. The definition of *strictly indecomposable to the left* is made analogously. If ϕ is strictly indecomposable to the right, then call ψ a *proper segment* of ϕ, if ϕ is strictly indecomposable to the right and $\phi = \psi + \theta$ where $\theta \neq 0$ or if ϕ is strictly indecomposable to the left and $\phi = \theta + \psi$ where $\theta \neq 0$. If ϕ is strictly indecomposable to the right or left, then write $\psi \ll \theta$ if ψ is embeddable in a proper segment of θ. With this notation, ϕ is Hagendorf, if ϕ is strictly indecomposable to the right, and for all $\psi < \phi$, it is true that $\psi \ll \phi$. In this case, write $\phi \in JH$.

Identify an ordinal with the set of its predecessors and with the order type of that set under ε. Then every additively indecomposable ordinal α is strictly indecomposable to the right, and has the further property that if $L \subseteq \alpha$ and α cannot be embedded in L, then L can be embedded in a proper initial segment of α.

At the time he told me of Hagendorf's problem, F. Galvin sketched for me the proof that all countable Hagendorf types are