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A CLASS OF MODIFIED ζ AND L-FUNCTIONS

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The purpose of this paper is the construction of a class of functions that have exactly the same complex zeros as the Riemann zeta function $\zeta(s)$, or as any Dirichlet function $L(s,\chi)$. The motivation for this construction is found in certain attempts to study the Riemann hypothesis.

The problem of the Riemann hypothesis has been approached, occasionally

(a) by attempts to study functions that share with $\zeta(s)(s=\sigma+it)$ certain analytic properties (e.g., representation by Dirichlet series, functional equation, Euler product, etc.), in order to see what restrictions these properties impose upon their zeros; or

(b) by the construction of functions, whose zeros are subject to certain restrictions (e.g., they do, or they don't satisfy a "Riemann hypothesis"), in the hope to detect similarities with, or differences from $\zeta(s)$, or $L(s, \chi)$.

To the first approach belong attempts to show that some of those analytic properties suffice to impose some kind of "Riemann hypothesis." These attempts were not too successful, in part because it turned out that functions like the the Epstein zeta function, with many of the properties of $\zeta(s)$ (functional equation, Dirichlet series-but no Euler product) may have zeros with $\sigma > 1$, and also with $0 < s < 1$ (see [4]).

To the second approach belongs, e.g., an attempt by Rademacher [5] to disprove the Riemann hypothesis, by studying the class of functions for which, assuming "the Riemann hypothesis," the sum $\sum_\gamma \gamma^{-1}\sin\gamma t = f(t)$ ($\gamma =$ imaginary part of the complex zero $1/2 + i\gamma$) exhibits the discontinuities known to occur when $\rho = 1/2 + i\gamma$ are zeros of $\zeta(s)$. It was shown, however, by Rubel and Straus (see [6] and [7]) that the known behaviour of $f(t)$ for $\zeta(s)$ is implied already by conditions much weaker than the Riemann hypothesis.

The results of the present paper seem to indicate that the last approach is unlikely to lead to interesting conclusions, but suggest a new and potentially useful approach. Indeed, we construct a class of functions of analytic character very different from that of $\zeta(s)$ and, nevertheless, with the same complex zeros. We also sketch the constrution of functions that share their complex zeros with Dirichlet's L-functions. In principle, the construction is valid for all Dirichlet series with an Euler product. These new functions have a Dirichlet