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SYMMETRIC DIFFERENCE IN ABELIAN GROUPS

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A groupoid $\mathfrak{A} = \langle A; *\rangle$ is called a left (resp. right) difference group if there is a binary operation $+$ in A such that the system $\langle A; +\rangle$ is an abelian group and $x * y = -x + y$ (resp. $x * y = x - y$). A symmetric difference group is a groupoid satisfying all the identities common to both left and right difference groups. In this note we determine the structure of a symmetric difference group. Using this, we show that any finitely based equational theory of symmetric difference groups is one-based. This includes the known result that the theories of left and right difference groups are one-based. Other known results on finitely based theories of rings also follow.

Let I (resp. J) stand for the class of all binary identities true in all left (resp. right) difference groups. Then the equational class of groupoids satisfying all the identities $I \cap J$ is, by definition, the class of all symmetric difference groups (SD-groups, for brevity). For example, the identity

$$(1) \qquad (x * y) * (((x * z) * (u * u)) * y) = z$$

belongs to the class $I \cap J$ and hence valid in every SD-group. To see this we only have to check whether (1) is true when $x * y = -x + y$ or $x * y = x - y$ in abelian groups. This is indeed the case. The main result of this paper is that the converse of the above statement is true, namely, any groupoid satisfying identity (1) is, in fact, an SD-group (Theorem 2). This is obtained through a structure theorem for groupoids satisfying identity (1).

1. The structure theorem.

We prepare the proof of the structure theorem with a lemma.

LEMMA 1. *Let $\mathfrak{A} = \langle A; F\rangle$ be an algebra having a binary polynomial $*$ and let w be a polynomial of the type of $\mathfrak{A}$. Then the algebra satisfies the identity*

$$(2) \qquad (x * y) * (((x * z) * w) * y) = z$$

iff $\mathfrak{A}$ has an abelian group reduct $\langle A; +, -, 0\rangle$ and there exists a map $\alpha\colon A \to A$ such that

(i) *$\mathfrak{A}$ satisfies the identity $w = 0$;*

(ii) *α is an involutoric[1] endomorphism of the group; and*

[1] We call a map $\alpha\colon A \to A$ involutoric if α^2 is the identity map on A.