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A SIMPLE PROOF OF THE EXISTENCE OF MODULAR AUTOMORPHISMS IN APPROXIMATELY FINITE DIMENSIONAL VON NEUMANN ALGEBRAS

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An elementary direct proof of Tomita–Takesaki Theorem for an AFD von Neumann Algebra.

1. Introduction. After that M. Tomita [5] proposed the existence of the modular automorphisms several proofs of Tomita–Takesaki theorem have been given by Takesaki, van Daele, Haagerup (unpublished) and Zsido [4, 6, 7, 8], but none of these is elementary. However a simple proof of the theorem for approximately finite dimensional von Neumann algebras (with a cyclic separating vector) may be extracted by an article of N.M. Hugenholtz and J.D. Wieringa 1], which was published very soon after the appearance of Tomita's original preprint. Motivated by the great interest that approximately finite dimensional von Neumann algebras have in Mathematics and in Physics, we present a simplified shorter version of the proof of Hugenholtz and Wieringa.

2. Statement and Proof. Let $\mathcal{R}$ be a von Neumann algebra acting on the Hilbert space $\mathcal{H}$ and $\xi \in \mathcal{H}$ a cyclic separating vector for $\mathcal{R}$ and then also for its commutant $\mathcal{R}'$. As usual we introduce the antilinear operators

$$S_0: A\xi, A \in \mathcal{R}, \rightarrow A^*\xi, \quad \mathcal{D}(S_0) = \mathcal{R}\xi,$$

$$F_0: B\xi, B \in \mathcal{R}', \rightarrow B^*\xi, \quad \mathcal{D}(F_0) = \mathcal{R}'\xi;$$

S_0 (and F_0) is a closable operator: in fact if $A \in \mathcal{R}$ and $B \in \mathcal{R}'$

$$\begin{aligned}(S_0A\xi, B\xi) &= (A^*\xi, B\xi) = (\xi, AB\xi) = (\xi, BA\xi)\\ &= (B^*\xi, A\xi) = (F_0B\xi, A\xi)\end{aligned}$$

so that $S_0^* \supset F_0$ and $\mathcal{D}(S_0^*)$ is dense.

In what follows we call $F = S_0^*$ the adjoint of S_0, $S = F^*$ the closure of S_0 and $\Delta = FS$ the modular operator which is non singular and positive. For the moment we suppose $\mathcal{R}$ finite dimensional; then there exists a faithful tracial state τ and for each state ω of $\mathcal{R}$ there exists a positive operator $H \in \mathcal{R}$ *s.t.*