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PROJECTIVE LATTICES

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This paper gives necessary and sufficient conditions for a lattice to be projective. The conditions are the Whitman condition, and a condition of Jónsson, and two new conditions explained below.

We begin with several definitions. First, a lattice L is projective if for any lattices M, N and any lattice homomorphism h of L into N and f of M onto N, there is a homomorphism g of L into M such that $f(g(a)) = h(a)$ for all $a \in L$. This is equivalent to the condition that there is a homomorphism f of a free lattice $FL(X)$ onto L and a homomorphism g of L into $FL(X)$ such that $f(g(a)) = a$ for all $a \in L$. The map $f \circ g$ is a *retraction*, i.e., it is an endomorphism of $FL(X)$ which is pointwise-fixed on its image. The image of a retraction is called a *retract*. Thus the class of projective lattices coincides with the class of retracts of free lattices. Since lattice epimorphisms, in the categorical sense, are precisely onto homomorphisms, [**11**, p. 150–151] the above definition of projective lattices agrees with the categorical one.

If a is an element of a lattice L and U is a finite, nonempty subset of L such that $a \leqq \vee U$, then U is a *cover* of a. A cover U of a is *trivial* if $a \leqq u$ for some $u \in U$. If U and V are finite nonempty subsets of L we write $U \ll V$ if for each $u \in U$ there is a $v \in V$ with $u \leqq v$. A cover U of a is called a *minimal cover* of a if whenever V is a cover of a such that $V \ll U$, then $U \subseteq V$. We let (W) denote the Whitman condition: for all a, b, c, d, $a \wedge b \leqq c \vee d$ implies either $a \wedge b \leqq c$, $a \wedge b \leqq d$, $a \leqq c \vee d$, or $b \leqq c \vee d$. Let f be a map from K onto L; then a map g from L into K is called a *transversal* of f if $f(g(a)) = a$ for all $a \in L$. L is a *bounded homomorphic image* of a free lattice if there is a homomorphism f mapping a free lattice, $FL(X)$, onto L such that $\{w \in FL(X): f(w) = a\}$ has a least and greatest element for each $a \in L$. This concept originated with R. McKenzie and a theorem of Jónsson, Kostinsky, and McKenzie [**16, 17**] states that a finitely generated lattice is projective if and only if it is a bounded homomorphic image of a free lattice and satisfies (W).

B. Jónsson, in his studies of sublattices of free lattices, defined $D_0(L)$ to be the set of join-prime elements of L ($a \in L$ is *join-prime* if $a \leqq b \vee c$ implies $a \leqq b$ or $a \leqq c$). $D_{k+1}(L)$ is the set of elements a of L such that if U is a nontrivial cover of a then there is a $V \subseteq D_k(L)$ such that V is a