PACIFIC JOURNAL OF MATHEMATICS
Vol. 75, No. 1, 1978

QUADRATIC FORMS WITH PRESCRIBED STIEFEL–WHITNEY INVARIANTS

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Milnor's construction of Stiefel–Whitney invariants for quadratic forms gives a map $\hat{w}$ from the Witt–Grothendieck ring of a field to a group arising in the K-theory of the field. Analogous maps are introduced here on the Witt ring and reduced Witt ring of the field. The images of these maps are studied. A central role is played by the degree of stability, in the sense of Elman and Lam, present in the Witt ring of the field.

In §1, we review Milnor's construction of $\hat{w}$ [**13**; also see 6] and show how it can be modified so as to give a well-defined map w on the Witt ring of a field. This construction systematizes and generalizes the way in which the determinant and Hasse symbol are modified to give the discriminant and Witt symbol. The problems of computing the images of w and $\hat{w}$ are equivalent. In §2, we show that $\hat{w}$ maps into an easily described subgroup k_{reg} of the target group of $\hat{w}$. Those fields with $\operatorname{Im}\hat{w} = k_{\mathrm{reg}}$ are shown in §3 to be precisely those with 3-stable Witt ring [**8**]. This is a special case of a fact about m-stability in the Witt ring for arbitrary m. Similar facts are established for w and for a map, w_{red}, which w induces on the reduced Witt ring. The exponent of the "cokernel" $k_{\mathrm{reg}}/\operatorname{Im}\hat{w}$ is studied in §4. If the Witt ring is n-stable, then the exponent is shown to be at most 2^f where $f = n - 1 + [-\log_2 n]$. (2^f equals the exponent for formally real algebraic function fields in n variables over the real numbers.) A similar estimate is given for fields of finite level. The exponent of the cokernel of w_{red} is computed explicitly. In §5 we provide examples of stability in Witt rings and reduced Witt rings. Particular attention is paid to certain familiar classes of algebraic function fields and Henselian valued fields. Finally, §6 is devoted to computing $\operatorname{Im}\hat{w}$ for superpythagorean fields. We hope this computation will be relevant to the computation of $\operatorname{Im} w_{\mathrm{red}}$ for all fields [**4**].

Throughout this paper F will denote a field not of characteristic two. Our notation closely follows that of Lam and Milnor [**12**; **14**]. (It will, however, be convenient for us to write "$\hat{w}$" in place of Milnor's "w".) Thus we denote the semigroup of equivalence (i.e., isometry) classes of nonsingular quadratic forms by $M(F)$, the Witt–Grothendieck ring by $\hat{W}(F)$, the Witt ring by $W(F)$, the torsion subgroup of $W(F)$ by $W_t(F)$, the reduced Witt ring by $W_{\mathrm{red}}(F)$, and the augmentation ideals of