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ALGEBRAS WHICH SATISFY A SECOND ORDER LINEAR PARTIAL DIFFERENTIAL EQUATION

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Let A be an algebra of complex valued functions satisfying a second order linear partial differential equation in a plane domain. If the equation is hyperbolic or parabolic, the functions of A are locally functions of only one variable. If the equation is elliptic, there exists a unique complex function λ such that $f_x = \lambda f_y$ for each f in A, and after a change of variables each function in A is analytic. If an algebra of functions satisfies the maximum principle, and one nonconstant function and its square satisfy an elliptic equation, then every function in the algebra satisfies this equation.

1. Introduction. In this paper we study algebras of complex valued functions defined on a plane domain, which satisfy some linear second order partial differenital equation

$$Lw = aw_{xx} + 2bw_{xy} + cw_{yy} + dw_x + ew_y = 0, \tag{1}$$

with real coefficients. We start with an example which turns out to be typical of the significant cases.

Let L be a self-adjoint elliptic operator:

$$Lw = \frac{\partial}{\partial x}(aw_x + bw_y) + \frac{\partial}{\partial y}(bw_x + cw_y), \tag{2}$$

where a, b, c are C^2 real functions on a simply connected domain, satisfying the normalizing condition $ac - b^2 = 1$. For each C^2 function u satisfying $Lu = 0$, we define (up to an additive constant) a conjugate function v by

$$v(x, y) = \int^{(x,y)} -(bu_x + cu_y)dx + (au_x + bu_y)dy. \tag{3}$$

It is easy to check the following facts: $Lv = 0$; the conjugate of v is $-u$; the set of functions $u + iv$ is an algebra; $(u + iv)^{-1}$ is in the algebra if $u + iv \neq 0$.

The functions $u + iv$ turn out to be analytic after the appropriate change of variables. Moreover, the example illustrates the only way