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CONSTRUCTION OF GENERALIZED NORMAL NUMBERS

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Let x be a real number, $0 \leqq x < 1$, and let $0.x_1x_2 \cdots$ be its expansion in the base B. Let $N(b, n)$ be the number of occurrences of the digit b in x up to x_n. Then x is called *digit normal* (in the base B) if

$$\lim_{n\to\infty} \frac{N(b, n)}{n} = \frac{1}{B}$$

for each of the B possible values of b. Let γ be any fixed B-ary sequence of length L and $N(\gamma, n)$ be the number of indices k for which $x_kx_{k+1} \cdots x_{k+L-1}$ is γ, that is, $N(\gamma, n)$ is the number of times γ appears in the first n digits of x. Then x is *normal* (in the base B) if

$$\lim_{n\to\infty} \frac{N(\gamma, n)}{n} = B^{-L}$$

for each of the B^L possible sequences γ, and B^{-L} is called the limiting frequency of γ in x.

The purpose of this paper is to construct a generalized normal number (in the base 2) in which these frequencies are weighted. For example, we will obtain infinite binary decimals in which the limiting frequency of occurrence of ones is 1/3 (in general, $p < 1$) rather than 1/2; consequently, any binary string γ of length L will have limiting frequency

$$(1/3)^K(2/3)^{L-K}$$

where K is the number of ones in γ.

For simplicity, the construction will be in the base 2, since generalization to the integer base B is straightforward and need be only briefly described.

Borel [1] proved that almost every number x, relative to Lebesque measure, is normal in the base 10. The simplest construction of a normal number is due to Champernowne [2] who showed that if the natural numbers are arranged in increasing order, the resulting decimal

$$0.12345678910111213 \cdots$$

is normal in the base 10. Copeland and Erdös [3] showed that certain subsequences of the natural numbers, arranged as above, are also normal; in particular for the sequence of primes,

$$0.13571113 \cdots$$