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DEFINABILITY IN THE LATTICE OF RING VARIETIES

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A variety of associative rings B immediately covers its subvariety A if every member of B outside A generates B. The variety $\{2x = 0, xy = 0\}$ is the unique equationally complete variety with precisely two immediate covers in the lattice of all associative ring varieties. The variety of all Boolean rings is first order definable in the lattice of all associative ring varieties. So are the varieties defined by $\{2x = 0, xy - yx = 0\}$ and $\{xy = 0\}$.

The set of all varieties of associative rings (not necessarily with 1) is a complete lattice under class inclusion. The variety of all Boolean rings is a member of this lattice. In contrast to the several equivalent axiomatizations of the theory of Boolean rings that can be found in the literature, there does not seem to exist any description of the Bloolean variety as a member of the lattice of all ring varieties. In this paper, a lattice theoretic notation is defined: b *immediately covers* a if $x \leqq a$ is equivalent to $x < b$. This notion is applied to show that the varieties defined by $\{2x = 0, xy = 0\}$, $\{xy = 0\}$, $\{2x = 0, xy - yx = 0\}$ and the variety of Boolean rings are all first order definable members in the lattice of all associative ring varieties.

In [5], Ralph McKenzie says that the variety of Boolean algebras is a definable member in the lattice of all varieties of universal algebras with two binary and one unary operation. The methods of [5] do not yield our results. In the present paper, the variety of Boolean rings with 1 is not defined as a member of the lattice of all varieties of associative rings with 1, neither is the variety of Boolean rings as a member of all varieties of commutative rings.

1. It is clear that b immediately covers a if and only if $a < b$ and a is the join of all elements of L strictly below b. Equivalently: b is an immediate cover of a if and only if b is completely join irreducible and a is the join of all elements of L that are strictly less than b. Thus, an element may have more than one immediate cover—the atoms of L are the immediate covers of the least element 0 of L. On the other hand, an element can immediately cover no more than one element.

Let A be a subset of L. A is called *first order definable* in L if there is a first order sentence $F(x)$ in one free variable x, relative