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ON THE METRIC THEORY OF DIOPHANTINE APPROXIMATION

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A conjecture of Duffin and Schaeffer states that

$$\sum_{n=2}^{\infty} \alpha_n \varphi(n) n^{-1} = +\infty$$

is a necessary and sufficient condition that for almost all real x there are infinitely many positive integers n which satisfy $|x - a/n| < \alpha_n n^{-1}$ with $(a, n) = 1$. The necessity of the condition is well known. We prove that the condition is also sufficient if $\alpha_n = O(n^{-1})$.

1. **Introduction.** Let $\{\alpha_n\}$, $n = 2, 3, 4, \cdots$, be a sequence of real numbers satisfying $0 \leqq \alpha_n \leqq 1/2$. We consider the problem of determining a sufficient condition on the sequence $\{\alpha_n\}$ so that for almost all real x the inequality

$$\left| x - \frac{a}{n} \right| < \frac{\alpha_n}{n} \tag{1.1}$$

holds for infinitely many pairs of relatively prime integers a and n. We note that there is no loss of generality if we restrict x to the interval $I = [0, 1]$. Let λ be Lebesgue measure on I and define

$$E_n = \bigcup_{\substack{a=1 \\ (a,n)=1}}^{n} \left(\frac{a - \alpha_n}{n}, \frac{a + \alpha_n}{n} \right),$$

where (a, n) denotes the greatest common divisor of a and n. Then our problem is to determine a sufficient condition on $\{\alpha_n\}$ so that

$$\lim_{N \to \infty} \lambda \left\{ \bigcup_{n=N}^{\infty} E_n \right\} = 1 . \tag{1.2}$$

It is clear that $\lambda(E_n) = 2\alpha_n \varphi(n)/n$ where φ is Euler's function. Thus by the Borel-Cantelli lemma,

$$\sum_{n=2}^{\infty} \lambda(E_n) = 2 \sum_{n=2}^{\infty} \frac{\alpha_n \varphi(n)}{n} = +\infty \tag{1.3}$$

is a *necessary* condition for (1.2) It has been conjectured by Duffin and Schaeffer [4] that (1.3) is also a sufficient condition for (1.2), but this has never been proved. Khintchine [7] showed that if $n\alpha_n$ is a decreasing function of n then (1.3) implies (1.2). (Actually, Khintchine's result is usually stated in a different but equivalent