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ONLY TRIVIAL BOREL MEASURES ON S_∞ ARE QUASI-INVARIANT UNDER AUTOMORPHISMS

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Let S_∞ be the group of all permutations of the integers. Then the only σ-finite Borel measures on S_∞ which are quasi-invariant under automorphisms are supported on the finite permutations.

1. Introduction. S_∞ is a complete separable metrizable group with the topology of pointwise convergence on the integers. S_∞ is not locally compact with this topology, and hence there is no σ-finite Borel measure on S_∞ which is invariant under left translations. For if there were such a measure, then there would be a locally compact group topology with a countable basis on S_∞ whose Borel structure coincides with the usual Borel structure (Theorem 7.1, Mackey [7]). This is a contradiction since the Borel structure of a complete separable metric group uniquely determines its topology. In fact, Mackey's result shows that there is no σ-finite Borel measure on S_∞ which is quasi-invariant under left translations. (Recall that a Borel measure μ on a Borel space X is said to be quasi-invariant under a group of Borel automorphisms G if μ and each of its translates under elements of G have precisely the same null sets.) However, even if G is a complete separable metric group which is not locally compact, then there may well be many Borel measures on G which are quasi-invariant under inner automorphisms. For example, let G be any Banach Space. Since G is abelian, any measure on G is invariant under inner automorphisms. The purpose of this paper is to prove the following theorem. It answers a generalization of a question posed by S. M. Ulam, and shows that the above phenomena cannot occur for S_∞. It roughly states that the inner automorphism action on S_∞ is so rich that some natural structures are precluded. This is a common occurence for S_∞. For example, Schreier and Ulam [8] have shown that every automorphism of S_∞ is inner, and Kallman [3] noted that S_∞ has a unique topology in which it is a complete separable metric group.

THEOREM 1.1. *The only σ-finite Borel measures on S_∞ which are quasi-invariant under automorphisms are supported by the finite permutations.*

A result of A. Lieberman [4] will be the main tool used to prove