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THE NON-ORIENTABLE GENUS OF THE n-CUBE

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For the purposes of embedding theory, a graph consists of a collection of points, called vertices, certain pairs of which are joined by homeomorphs of the unit interval, called edges. Edges may intersect only at vertices, and no vertex is contained in the interior of an edge. The graph thus becomes a topological space as a subspace of R^3. An embedding of a graph G in a compact 2-manifold (surface) S is then just an embedding of G in S as a topological space. The genus, $\gamma(G)$, of G is the minimum genus among all orientable surfaces into which G may be embedded. The non-orientable genus, $\tilde{\gamma}(G)$, is defined analogously. The n-cube, Q_n, is a well known graph wnich generalizes the square and the standard cube. In this paper the following formula is proven:

THEOREM.

$$\tilde{\gamma}(Q_n) = \begin{cases} 2 + 2^{n-2}(n-4) & n \geqq 6 \\ 3 + 2^{n-2}(n-4) & n = 4, 5 \\ 1 & n \leqq 3 . \end{cases}$$

Introduction. In 1955, Ringel [3] showed that the orientable genus of the n-cube is given by

$$(1) \qquad \gamma(Q_n) = 1 + 2^{n-3}(n-4) .$$

This result was also obtained independently by Beinecke and Harary [1]. Since then, genus formulae, both orientable and non-orientable, have been obtained for several classes of graphs, including the complete bipartite graph, the octahedral graphs and many of the other multipartite graphs (see, for example [2]). As a result, the n-cube is perhaps the best known graph for which a genus question has remained open. The present paper fills this gap.

Preliminaries. Let Z_2^n be the elementary abelian 2-group of rank n, $Z_2 X \cdots X Z_2$. If $x \in Z_2^n$, let $[x]_k$ be the kth ordinate of x, so that $x = ([x]_1, \cdots, [x]_n)$. Let $1_k \in Z_2^n$ be the element such that $[1_k]_m = 1$ iff $m = k$, and let $\Delta_n = \{1_k \in Z_2^n \mid 1 \leqq k \leqq n\}$. Then the n-cube, Q_n, is the Cayley graph (Z_2^n, Δ_n); that is, Z_2^n is the vertex set of Q_n, and for $x, y \in Z_2^n$, $\{x, y\}$ is an edge iff $x + y \in \Delta_n$. If $\{x, y\}$ is an edge, it is said to have the *color* $x + y$, and is called a $(x + y)$-edge.

Let $(x; c_1, \cdots, c_m)$, where $x \in Z_2^n$ and $c_i \in \Delta_n$, denote the walk x, $x + c_1$, $x + c_1 + c_2, \cdots, x + c_1 + \cdots + c_m$ in Q_n. Note that any walk $x_0, \cdots, x_m$