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ISOTOXAL TILINGS

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A plane tiling $\mathscr{T}$ is called *isotoxal* provided the group of symmetries of $\mathscr{T}$ acts transitively on the arcs (edges) of $\mathscr{T}$. A method for classifying isotoxal tilings is described, and the number of types of normal isotoxal tilings is determined.

1. A plane tiling $\mathscr{T} = \{T_i \mid i = 1, 2, \cdots\}$ is called *isohedral* if its symmetry group $S(\mathscr{T})$ is transitive on the tiles T_i of $\mathscr{T}$, *isogonal* if $S(\mathscr{T})$ is transitive on the nodes (vertices) of $\mathscr{T}$, and *isotoxal* if $S(\mathscr{T})$ is transitive on the arcs (edges) of $\mathscr{T}$. (The word "isotoxal" is derived from the Greek "τοξον" meaning "arc".) In recent papers [2], [3] we have enumerated the different types of isohedral and isogonal tilings of the plane. Here we prove the following theorem.

THEOREM 1. *There exist* 26 *types of normal isotoxal tilings of the plane. Of these,* 25 *are also either isohedral, or isogonal, or both.*

By a "normal" tiling we mean one which is *bounded* (that is, the tiles are uniformly bounded in diameter and their inradii are uniformly bounded away from zero) and for which the intersection of any two tiles is either empty, or an edge, or a vertex of each. We thus exclude vertices of valence two, and digons. (As in [2] and [3], a *vertex* is any point common to at least three tiles, and a tile is called an *n-gon* if it has n vertices and n sides. Here *a side* of a tile is the part of its boundary between consecutive vertices, and the word n-gon must not be taken to imply that the tile is convex or even that its sides are straight line segments.)

The proof of Theorem 1 will be given in the next section. It proceeds in two stages. First we determine the "combinatorial isotoxal tilings" and establish that there are exactly 30 types of these. Secondly we show that 26 of these types can be realized by actual tilings. Thus it will be seen that the procedure is analogous to that used in enumerating the isohedral and isogonal tilings, though the detailed analysis is rather different.

The third section will deal with tilings which are not normal. Here the main result will be the following.

THEOREM 2. *There exist* 15 *types of bounded isotoxal tilings*