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ON ARC LENGTH SHARPENINGS

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This paper introduces two new sharpenings:

THEOREM. Let A denote a rectifiable arc (with length $l(A)$) of a metric space, let P denote a finite, normally-ordered subset of A, and let $l(T^*(P))$ denote the linear content of a mini-tree $T^*(P)$ spanning P. Then $\text{l.u.b.}_{P\subset A}\, l(T^*(P)) = l(A)$.

DEFINITION. If E is a nonempty subset of a set P that is spanned by tree T, then T is said to be on E.

THEOREM. Let $\sigma(E)$ denote the greatest lower bound of the linear contents of all trees on E. If A denotes a rectifiable arc of a finitely compact metric space, then $\text{l.u.b.}_{E\subset A}\, \sigma(E) = l(A)$, where E denotes any finite normally-ordered subset of A.

On arc length sharpenings.[1] It is convenient to call an unordered pair of distinct points p, q of a metric space M a *segment*, denoted by $\{p, q\}$. Each of the points p, q of the segment $\{p, q\}$ is an *endpoint* of the segment, and the *length* of $\{p, q\}$ is the distance pq of its endpoints.

A nonempty set S of distinct segments forms a *chain* C provided the end points of the segments may be labelled $a_0, a_1, \cdots, a_k$ (with all the a_i's representing pairwise distinct elements of M) so that the elements of S are $\{a_0, a_1\}, \{a_1, a_2\}, \cdots, \{a_{k-1}, a_k\}$. The chain is said to *join* a_0 and a_k; the points $a_0, a_1, \cdots, a_k$ are the *vertices* of the chain.

A nonempty set S of segments forms a *tree* T provided each two distinct points of the set of endpoints of the segments are joined by exactly one chain of its segments. The *vertices* of T are the endpoints of its segments. The segments of a tree are called *branches*, and the *linear content* of a tree is the sum of the lengths of its branches. If a tree T has set E as its vertex set, then T is said to *span* E. If E is a nonempty subset of a set P, and tree T spans P, then T is said to be *on* E.

A finite subset E (containing at least two points) of M is spanned by only a finite number of trees. Let $L(E)$ denote the minimum of the linear contents of the trees that span E and let $T^*(E)$ symbolize any tree spanning E whose linear content $l(T^*(E))$ equals $L(E)$. $T^*(E)$ is referred to as a *mini-tree spanning* E.

Denote by $\sigma(E)$ the greatest lower bound of linear contents of all trees that span P where $P \supset E$ (P is a finite subset of M); that

[1] From research for University of Missouri Dissertation (1973).