PACIFIC JOURNAL OF MATHEMATICS
Vol. 76, No. 2, 1978

UNIFORM REPRESENTATIONS OF CONGRUENCE SCHEMES

JOEL BERMAN AND GEORGE GRÄTZER

A congruence scheme Σ is a finite sequence of polynomials. A nontrivial equational class K is a representation of Σ iff the principal congruences in K can be described in a natural fashion by Σ. In this paper it is shown that a necessary and sufficient condition for a congruence scheme Σ whose polynomials do not contain constants to have a representation is that each polynomial in the sequence be at least binary.

1. Introduction. For an algebra $\mathfrak{A}$ and $a, b \in A$, let $\Theta(a, b)$ denote the smallest congruence relation under which $a \equiv b$; such relations are called *principal*. For instance, in the class $\boldsymbol{D}$ of distributive lattices (see [1]):

$$c \equiv d(\Theta(a, b)) \text{ iff}$$
$$\begin{aligned} c &= p_0(b, a, b, c, d) \\ p_0(a, a, b, c, d) &= p_1(a, a, b, c, d) \\ p_1(b, a, b, c, d) &= p_2(b, a, b, c, d) \\ p_2(a, a, b, c, d) &= p_3(a, a, b, c, d) \\ p_3(b, a, b, c, d) &= d\ , \end{aligned}$$

where

$$\begin{aligned} p_0(x, y_0, y_1, y_2, y_3) &= ((x \wedge y_0) \vee y_2) \wedge (y_2 \vee y_3)\ , \\ p_1(x, y_0, y_1, y_2, y_3) &= (x \vee y_0 \vee y_2) \wedge (y_2 \vee y_3)\ , \\ p_2(x, y_0, y_1, y_2, y_3) &= ((x \vee y_0) \wedge y_2) \vee y_3\ , \\ p_3(x, y_0, y_1, y_2, y_3) &= (x \wedge y_0 \wedge y_2) \vee y_3\ . \end{aligned}$$

This is one example of a congruence scheme (for a general definition, see §2). The general definition of a congruence scheme permits an arbitrary sequence $p_0, \cdots, p_{n-1}$ of polynomials and the polynomials may have any number of variables.

As the simplest example of a congruence scheme, let $\boldsymbol{K}$ be an equational class and let us assume that for all $\mathfrak{A} \in \boldsymbol{K}$ the following holds:

$c \equiv d(\Theta(a, b))$ iff $c = a + e$ and $d = b + e$ for some $e \in A$, where $+$ is a binary operation of $\boldsymbol{K}$.

Congruence schemes have been investigated in [5] under the name 1-good systems and in [1].