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TRIGONOMETRIC APPROXIMATION THEORY IN COMPACT TOTALLY DISCONNECTED GROUPS

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We study some aspects of the problem of approximating functions on a compact totally disconnected group by trigonometric polynomials. In the classical case, approximation is connected with smoothness properties. By appropriately defining smoothness, one obtains this connection in the totally disconnected case also, and there are analogues of the classical results.

In particular it is shown that the Lipschitz class to which a function belongs can be identified by the best approximation characteristics of the function by trigonometric polynomials (Theorems 2 and 3), and that functions which are easily approximated by trigonometric polynomials have absolutely convergent Fourier series (Theorems 1 and 4).

Let G be a compact group. Let σ be an equivalence class of continuous irreducible unitary representations of G. The set of all such σ is called the dual object of G and is denoted by Σ. For each $\sigma \in \Sigma$ fix a $U^{(\sigma)} \in \sigma$ and let H_σ be the Hilbert space in which $U^{(\sigma)}$ acts. The dimension of H_σ is denoted by d_σ. This notation is consistant with the notation in the book of Hewitt and Ross [5]. Any unexplained notation may be found there. We now restrict our attention to infinite compact totally disconnected groups whose dual objects are countable or equivalently, which have a countable neighborhood base $G = G_0 \supset G_1 \supset \cdots$ at the identity e consisting of open (hence closed) normal subgroups [7, p. 132]. Since these subgroups are open, they have positive Haar measure, and since each coset of a given subgroup has the same Haar measure, it follows that the index of G_{n+1} in G_n is $m(G_n)/m(G_{n+1})$, where m denotes the normalized Haar measure on G. We say that a bounded function is in the Lipschitz class of order $\alpha > 0$ (with respect to a neighborhood system $\{G_n\}$) if

$$\sup_{x \in G_k} \|f(x\cdot) - f(\cdot)\|_\infty \leqq Cm(G_k)^\alpha$$

where C is some constant independent of k.

This definition involved left translation. That right translation gives the same class of functions can be seen as follows. We have $|f(xy) - f(y)| = |f(yy^{-1}xy) - f(y)|$, and since G_k is a normal subgroup, $y^{-1}xy$ runs through G_k as x runs through G_k.

The Lipschitz class of order α will be denoted by $\mathrm{Lip}_\alpha(G)$ and