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HOMEOMORPHIC MEASURES IN THE HILBERT CUBE

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A Borel probability measure μ in the Hilbert cube is homeomorphic to the usual product measure if and only if it is positive for nonempty open sets and zero for points. The transformation can be effected by a homeomorphism equal to the identity on any prescribed μ-null Z-set. Several extension, approximation, and embedding theorems are obtained as applications.

1. Introduction and basic theorem. Let $Q = I^\infty = \prod_1^\infty I_i$ denote the Hilbert cube, where each $I_i = [0, 1]$, and let λ denote Lebesgue product measure in Q restricted to Borel sets. If h is any homeomorphism of Q onto itself, then $E \mapsto \lambda(h(E))$ defines a Borel measure in Q which we shall denote by λh. Any measure μ that admits such a representation is said to be *homeomorphic* (or *topologically equivalent*) to λ. Evidently any such measure in Q is a normalized Borel measure and it must be zero for points (nonatomic) and positive for nonempty open sets (locally positive). Our aim is to prove the converse of this statement, with appropriate refinements specifying what kinds of subsets can be kept fixed, and to give several applications of this result: an extension theorem for measure preserving homeomorphisms between subsets of Q, a version of Luzin's theorem for measure preserving homeomorphisms, and a characterization of the topological measure spaces that can be embedded in Q by a measure preserving homeomorphism. The potential usefulness of such a theorem in connection with the last mentioned problem has long been recognized, especially by Dorothy Maharam and A. H. Stone (cf. [9] and [10]). We include also a couple of related results which do not depend on the basic theorem: a characterization of sets topologically equivalent to nullsets, and a version of Goffman's approximation theorem for one-one measurable transformations, both of which were previously known only in the finite-dimensional case.

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It is known [7] (see also [4]) that to characterize measures homeomorphic to Lebesgue measure in I^n (n finite) it is necessary to require that $\mu(\partial I^n) = 0$ in addition to the requirements mentioned above. Since I^∞ has no boundary it is not surprising that this condition can be dropped but it is remarkable that no other condition is needed to replace it.

Let Q be given the metric $d(x, y) = \sum_1^\infty |x_i - y_i|/2^i$. By an *r-set*