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A GENERAL COINCIDENCE THEORY

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Consider two topological spaces X and Y, two maps $f, g: X \to Y$, and a relation R on $Y, R \subset Y \times Y$. A Lefschetz-type theorem is established with regard to the existence of an $x \in X$ such that $g(x)Rf(x)$. Several ideas related to the Lefschetz Fixed Point Theorem, such as periodic point theorems, local fixed point index, asymptotic fixed point and periodic point theorems, are carried over to this more general situation.

1. Introduction. Consider two topological spaces X and Y, two maps $f, g: X \to Y$, and a relation R on $Y, R \subset Y \times Y$. Under what circumstances must there exist an $x \in X$ such that $g(x)Rf(x)$, i.e., $(g(x), f(x)) \in R$. In this paper we do three things with respect to this problem. First, we show how to obtain related homology maps from $H_*(Y)$ into itself. Each of these homology maps h is such that $\Lambda(h) = \sum_{n \geq 0} (-1)^n$ trace $h_n \neq 0$ implies $g(x)Rf(x)$ for some $x \in X$. Secondly, we find an interpretation in terms of f and g for the condition $\Lambda(h^r) \neq 0$, where $h^r = h \circ h \circ \cdots \circ h$, r times. We show that if $\Lambda(h^r) \neq 0$, then there exist points $x_1, x_2, \cdots, x_r \in X$ such that $g(x_{i+1})Rf(x_i)$ for $1 \leqq i < r$ and $g(x_1)Rf(x_r)$. Thirdly, we show that there are many situations in which one can assert that $\Lambda(h^r) \neq 0$ for some $r \leqq N$, where N is determined by the situation at hand.

One application is the following apparently new theorem about spheres. Theorem 4.5: If $f, g: S^{2n} \to S^{2n}$ are continuous maps such that $f(A) \neq g(A)$ for all $A \subset S^{2n}$ with cardinality of $A = 1$ or 2, then both f and g are null homotopic. Another application concerns lines in the complex projective plane as follows. Remark 4.9: If L is a line in CP^2 and f is a continuous map from L into the space of lines in CP^2, then there exists two points $a, b \in L$ such that $f(a)$ goes through b and $f(b)$ goes through a.

Section 2 contains some notation and conventions. In § 3 we establish our central theorems. Section 4 contains applications. In § 5 we consider extensions to more than two functions. We outline a local index theory in § 6. In § 7 we establish some asymptotic theorems. In the last section, § 8, we discuss a very general situation involving four spaces, two maps, and two relations.

Several authors have dealt with the problems of fixed points and coincidence points by similar methods. Fuller [8], Fadell [7], Brown [3] and [4], and Roitberg [15] deal with fixed points and