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SURJECTIVITY RESULTS FOR ϕ-ACCRETIVE SET-VALUED MAPPINGS

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Let X and Y be Banach spaces. A mapping $f: X \to 2^Y$ is said to be locally strongly ϕ-accretive if for each $y_0 \in Y$ and $r > 0$ there exists a constant $c > 0$ such that if $z \in f(X) \cap B_r(y_0)$ and $x \in f^{-1}(z)$, then for all u sufficiently near x and $w \in f(u)$: $(\phi(x-u), (w-z)) \geqq c\|x-u\|^2$, where ϕ: $X \to Y^*$ is a suitably restricted mapping. A number of surjectivity results are obtained for this class of mappings, along with some other basic results.

In this continuation of the study of surjectivity results obtainable by application of refined versions of the fixed point approach of Caristi [5], we turn our attention to analogues of the results of Kirk [11] for set-valued mappings.

1. Preliminary results.

One of the applications we have in mind requires the following minor reformulation of the fixed point theorem of Downing-Kirk [7]. (Recall that a mapping $f: X \to Y$, where X and Y are metric spaces, is said to be *closed* if the conditions $x_n \to x$, $x_n \in X$, and $f(x_n) \to y$ imply $f(x) = y$.)

THEOREM 1. *Let (X, d_1) and (Y, d_2) be complete metric spaces, $g: X \to X$ an arbitrary mapping, and $f: X \to Y$ a closed mapping. Suppose there exists a closed subset S of Y for which $f^{-1}(S) \neq \emptyset$ and $g: f^{-1}(S) \to f^{-1}(S)$, and suppose there exists a lower semicontinuous mapping $\varphi: f(X) \cap S \to R^+$ (the nonnegative reals) and a constant $c > 0$ such that*

$$\max\{d_1(x, g(x)), cd_2(f(x), f(g(x)))\} \leqq \varphi(f(x)) - \varphi(f(g(x))), \ x \in f^{-1}(S).$$

Then g has a fixed point in X.

The above theorem reduces to the theorem of Downing-Kirk [7] if $S = Y$ and to the original formulation of Caristi [5] if $S = Y = X$ and f is the identity. Indeed, because f is a closed mapping, the set $f^{-1}(S)$ is complete relative to the metric ρ defined by

$$\rho(u, v) = \max\{d_1(u, v), cd_2(f(u), f(v))\}, \quad u, v \in f^{-1}(S).$$

Thus Theorem 1 is a direct consequence of the original version of Caristi's theorem applied to the space $(f^{-1}(S), \rho)$. This observation is due to W. L. Bynum.

We also remark that Caristi's original theorem is essentially