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CENTRAL MOMENTS FOR ARITHMETIC FUNCTIONS

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The only central moment considered in probabilistic number theory up until now has been the "variance" of an arithmetic function. This paper considers the case of higher central moments for such functions. It will be shown that if f is an additive complex valued arithmetic function then

$$\sum_{m\leq n}|f(m)-A(n)|^{2K}=O(n(\log\log n)^{2K-2}\sum_{p^\alpha\leq n}|f(p^\alpha)|^{2K}p^{-\alpha})$$

where K is a positive integer and

$$A(n)=\sum_{p^\alpha\leq n}f(p^\alpha)p^{-\alpha}\,.$$

It will also be shown that if f is an additive real valued arithmetic function and K is an odd positive integer, then

$$\sum_{m\leq n}(f(m)-A(n))^K=O(n(\log\log n)^{K-2+1/K}\sum_{p^\alpha\leq n}|f(p^\alpha)|^Kp^{-\alpha})\,.$$

1. Preliminaries. Given a fixed positive integer K let X be a K-tuple of prime powers p^α, where the primes need not be distinct. Y is defined similarly. Next we define

$$||X||=\operatorname{Max}\{p^\alpha\colon p^\alpha \text{ is a component of } X\}$$

and $|X|=\prod p^\alpha$ where the product is over those p^α which are components of X. By X_j we shall mean the j-tuple consisting of the first j components of X, and $\tilde{X}_j$ shall denote the $K-j$-tuple consisting of the last $K-j$ components of X. X_jY_k shall denote the first j components of X followed by the first k components of Y. By $X_j||m$ we shall mean that $p^\alpha||m$ for all the components of X_j. If f is an arithmetic function, then we define $F(X)$ to be $\prod f(p^\alpha)$ where the product is over all the components p^α of X.

LEMMA 1. *Given the M distinct prime powers $P_i=p_i^{\alpha_i}$, $i=1,\cdots,M$, and the positive integer n,*

$$W(M,n)=n^{-1}\sum_{\substack{k\leq n\\ P_i||k,\,i\leq M}}1=\prod_{i=1}^{M}P_i^{-1}(1-p_i^{-1})+O(n^{-1})$$

where $|O(n^{-1})|\leq(3\cdot 2^M-1)n^{-1}$.

Proof. Let $N=L\prod_{i=1}^M P_i$ for any positive integer L. We will now show by induction on M that for all such N

$$W(M,N)=\prod_{i=1}^{M}P_i^{-1}(1-p_i^{-1})+O(N^{-1}) \tag{1.1}$$