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ON JOINT NUMERICAL RANGES

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The joint numerical status of commuting bounded operators A_1 and A_2 on a Hilbert space is defined as $\{(\phi(A_1), \phi(A_2))$ such that ϕ is a state on the C^*-algebra generated by A_1 and $A_2\}$. It is shown that if A_1 and A_2 are commuting normal operators then their joint numerical status equals the closure of their joint numerical range. It is also shown that certain points in the boundary of the joint numerical range are joint approximate reducing eigenvalues.

The joint numerical range of A_1 and A_2 denoted by $w(A_1, A_2)$ is $\{((A_1x, x), (A_2x, x))$ such that $x \in H$ and $||x|| = 1\}$. Thus $w(A_1, A_2)$ is a bounded subset of C^2. It is not known whether this set is convex, Dash [4, 6]. In this note, we shall show that there is faithful $*$ representation of the C^*-algebra generated by A_1 and A_2, $C^*(A_1, A_2)$, under which the joint numerical range of A_1 and A_2 is convex. Following Berberian and Orland [1], we study the joint numerical status of A_1 and A_2, $\Sigma(A_1, A_2) = \{(\phi(A_1), \phi(A_2))$ such that ϕ is a state on $C^*(A_1, A_2)\}$. If A_1 and A_2 are commuting normal operators then $\Sigma(A_1, A_2) = \bar{w}(A_1, A_2)$. We also show that certain points in the boundary of $w(A_1, A_2)$ are joint approximate reducing eigenvalues.

For the sake of notational convenience, all the results are being stated for two commuting operators. However, the results hold for any finite family of commuting operators.

Let $B(H)$ denote the algebra of all bounded linear operators on the Hilbert space H. Let $C^*(A_1, A_2)$ denote the C^*-algebra generated by I, A_1, and A_2. Let Σ denote the set of all states on $C^*(A_1, A_2)$. Any state ϕ in Σ induces a representation Π_ϕ of $C^*(A_1, A_2)$ which acts on a Hilbert space H_ϕ and has a cannonical cyclic vector ξ_ϕ. Also any maximal left ideal of $C^*(A_1, A_2)$ is of the form $K(\psi)= \{A \in C^*(A_1, A_2)$ such that $\psi(A^*A) = 0\}$ for some pure state ψ on $C^*(A_1, A_2)$. For details concerning this the reader is referred to Dixmier [7]. The joint approximate point spectrum of A_1 and A_2, denoted by $a(A_1, A_2)$, is $\{(z_1, z_2)$ such that there exists a sequence $x_n \in H$, $||x_n|| = 1$ such that $||(A_1 - z_1)x_n|| \to 0$ and $||(A_2 - z_2)x_n|| \to 0\}$ which is the same as $\{(z_1, z_2)$ such that $B(H)(A_1 - z_1) + B(H)(A_2 - z_2) \neq B(H)\}$.

First we shall show that $a(A_1, A_2)$ depends only on the C^*-algebra generated by A_1 and A_2. Our proof is similar to Bunce [2].