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CAPACITIES OF COMPACT SETS IN LINEAR SUBSPACES OF R^n

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We consider two types of spaces, the Bessel potential spaces $L_\alpha^p(R^n)$ and the Besov spaces $\Lambda_\alpha^p(R^n)$, $\alpha > 0$, $1 < p < \infty$. Associated in a natural way with these spaces are classes of exceptional sets. We characterize the exceptional sets for $\Lambda_\alpha^p(R^n)$ by an extension property for continuous functions and prove an inequality between Bessel and Besov capacities.

The classes of exceptional sets for the spaces $L_\alpha^p(R^n)$ have been studied by the concept of capacity [5]. Capacity definitions of these classes are given in § 2.

Bessel potential spaces and Besov spaces in R^n and R^{n+1} are connected by restriction theorems. A short statement of these results is the following:

$$L_\beta^p(R^{n+1})|_{R^n} = \Lambda_\alpha^p(R^n) \tag{1.1}$$

$$\Lambda_\beta^p(R^{n+1})|_{R^n} = \Lambda_\alpha^p(R^n)\ , \tag{1.2}$$

where $\alpha>0$, $1<p<\infty$, and $\beta = \alpha + 1/p$. (O. V. Besov [4] and E.M. Stein [7].)

The restriction theorem above gives relations between exceptional classes of different spaces L_α^p and Λ_α^p in R^n and R^{n+1}.

This enables us to prove an extension theorem for continuous functions on a compact set $K \subset R^n$ into $\Lambda_\alpha^p(R^n)$ (Theorem 1) analogous to the $L_\alpha^p(R^n)$ — case contained in [6, Theorem 1]. Finally we prove an inequality between the capacities defining the classes of exceptional sets for $\Lambda_\alpha^p(R^n)$ and $L_\alpha^p(R^n)$ (Theorem 2).

2. Preliminaries and statements of the theorems. We consider the n-dimensional space R^n of n-tuples $x = (x_1, x_2, \cdots, x_n)$. Points in R^{n+1} are written (x, x_{n+1}), where $x \in R^n$ and $x_{n+1} \in R^1$. Then R^n is identified as the subspace $\{(x, 0); x \in R^n\}$ of R^{n+1}. Compact sets are denoted by K. If $K \subset R^n$ then K is a compact subset of R^{n+1} as well. As usual, the space of p-summable functions is denoted by $L^p(R^n)$ with norm $||\cdot||_p$. The Bessel kernel G_α^n in R^n is the $L^1(R^n)$-function whose Fourier transform equals $(1 + |x|^2)^{-\alpha/2}$, $\alpha > 0$.

The space of convolutions $U = G_\alpha^n * f$, where $f \in L^p(R^n)$, with the norm $||U||_{\alpha,p} = ||f||_p$, is denoted by $L_\alpha^p(R^n)$, $\alpha > 0$, $1 \leqq p < \infty$. A function $U \in \Lambda_\alpha^p(R^n)$, $1 \leqq p \leqq \infty$, $0 < \alpha < 1$ if

$$|U|_{\alpha,p} = ||U||_p + \left(\iint \frac{|U(x) - U(y)|^p}{|x - y|^{\alpha p+n}}\, dxdy\right)^{1/p}$$