PACIFIC JOURNAL OF MATHEMATICS
Vol. 78, No. 1, 1978

REMARK ON A PAPER OF STUX CONCERNING SQUAREFREE NUMBERS IN NON-LINEAR SEQUENCES

G. J. RIEGER

Stux studied squarefree numbers of the form $[f(n)]$; his most interesting application is $f(n)=n^c$ for real c with $1<c<4/3$. We would like to point out that a stronger result follows immediately from estimates of Deshouillers.

Let $1 < c < 2$, $x \geqq 1$; denote by $N_c(x; k, l)$ the number of natural numbers $n \leqq x$ with $[n^c] \equiv 1 \bmod k$. According to [1], we have

$$N_c(x; k, l) = \frac{x}{k} + O_c((x^{1+c}k^{-1})^{1/3}) \quad \text{for} \quad x^{c-5/4} \leqq k < x^{c-1/2}\,, \tag{1}$$

$$N_c(x; k, l) = \frac{x}{k} + O_c((x^{4+c}k^{-1})^{1/7}) \quad \text{for} \quad k < x^{c-5/4}\,. \tag{2}$$

Denote by $S_c(x)$ the number of squarefree numbers of the form $[n^c]$ with natural $n \leqq x$; the inclusion-exclusion principle in the form $|\mu(n)| = \sum_{d^2|n, d>0} \mu(d)$ gives

$$S_c(x) = \sum_{d^2 \leqq x^c} \mu(d) N_c(x; d^2, 0) \qquad (x \geqq 1)\,. \tag{3}$$

For $d^2 \geqq x^{c-1/2}$ we use the trivial estimate $N_c(x; d^2, 0) = O(x^c d^{-2})$; using

$$\sum_{d>t} d^{-2} = O(t^{-1}) \qquad (t \geqq 1)\,, \tag{4}$$

we obtain

$$S_c(x) = \sum_{d^2 < x^{c-1/2}} \mu(d) N_c(x; d^2, 0) + O(x^{(2c+1)/4})\,. \tag{5}$$

In case $c \leqq 5/4$, we use (1) and

$$\sum_{0<d\leqq t} d^{-2/3} = O(t^{1/3}) \qquad (t \geqq 1) \tag{6}$$

in (5); this gives

$$S_c(x) = \sum_{d^2 < x^{c-1/2}} \mu(d) d^{-2} x + O_c(x^{(2x+1)/4})\,. \tag{7}$$

In case $c > 5/4$, we split the sum in (5) according to $d^2 <$ or $\geqq x^{c-5/4}$ and apply (2) and (1); using $\sum_{0<d\leqq t} d^{-2/7} = O(t^{5/7})$ $(t \geqq 1)$ and (6), we obtain again (7). But (7), $\sum_{d>0} \mu(d) d^{-2} = 6\pi^{-2}$, and (4) give immediately