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PARTITIONING INTEGERS USING A FINITELY GENERATED SEMIGROUP

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Denoting by Γ the semigroup of positive integers generated by two fixed primes, let $r_k(N)$ be the number of partitions of N as a sum of k elements of Γ. Our main result is that $r_2(N)$ is a bounded function of N. Incidentally, we obtain an estimate of the number of distinct prime divisors of numbers of the form $1+q^n$. Boundedness of $r_k(N)$ would resolve an approximation theoretic conjecture of D. J. Newman.

Let Γ be a finitely generated semigroup of positive integers. For a positive integer N, let $r_k(N)$ be the number of partitions of N into k parts from Γ. Donald J. Newman has asked the following question:

Is $r_k(N)$ a bounded function of N, for all k?

This question arose in the context of a general problem of approximation theory; that is, the determination of when, for a given function $f(x)$, the functions $\{f(kx)\}_{k=-\infty}^{k=\infty}$ generate a dense subspace E_f of some function space. This problem has been considered by Neuwirth, Ginsberg and Newman in [3] for $f(x)$ a trigonometric polynomial. In his report to the Canterbury conference on complex analysis ([4], 1973), Newman stated a conjecture: Let $f(z) = z + a_2z^2 + \cdots + a_nz^n$ (here $z = e^{i\theta}$). To $f(z)$ we associate a "Dirichlet polynomial"

$$D(s) = 1 + a_2/2^s + \cdots + a_n/n^s .$$

Then E_f is dense in $L^p(1 \leqq p < \infty)$ if and only if $D(s)$ has no zeros in $\operatorname{Re} s > 0$, and E_f is dense in L^∞ if and only if $D(s)$ is bounded away from zero in $\operatorname{Re} s > 0$. Newman asserts that the settling of this conjecture for $p < \infty$ depends on making a connection between norms in the z and s variables, and that this connection can be made according to a classical result of Szidon, if the above number theoretic question has an affirmative answer.

In this paper we shall consider the simplest case of the question, when Γ is generated by two primes, and $k = 2$. A complete proof of Newman's conjecture for the corresponding $f(z)$ would require a proof for this Γ, for all k.

Let p, q be distinct primes; we shall denote by Γ the multiplicative semigroup of nonnegative integers generated by $\{0, p, q\}$. For any integer N, let $r_2(N)$ denote the number of representations

$$N = \alpha + \beta \qquad (\alpha, \beta \in \Gamma) .$$