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# ON RELATIONS FOR REPRESENTATIONS OF FINITE GROUPS

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**Let $G$ be a finite group, and suppose that**

$$A\colon G \longrightarrow \mathrm{GL}(n, C)$$

**is a (complex) representation of $G$ with character $\chi$. A (complex) linear relation for $A$ is a formal complex linear combination $\sum_{g\in G} a_g g$ such that $\sum_{g\in G} a_g A(g) = 0$.**

**We prove the following theorem, which determines the linear relations in terms of the character $\chi$.**

THEOREM. **Let $A$ be a representation for a finite group $G$, let $\chi$ be the character of $A$, and let $\{g_1, \cdots, g_k\}$ be a subset of $G$. Then $\sum_{j=1}^{k} a_j g_j$ is a relation for $A$ if and only if $\sum_{j=1}^{k} \chi(g_i g_j^{-1}) a_j = 0$, for all $i = 1, \cdots, k$.**

NOTE 1. If $C$ is the $k \times k$ matrix whose $ij$-entry is $\chi(g_i g_j^{-1})$ and $a$ is the column vector whose $j$th entry is $a_j$, then the above conclusion can be rephrased as follows:

$$\sum_{j=1}^{k} a_j g_j \quad \text{is a relation for } A \iff Ca = 0 \;.$$

NOTE 2. The above theorem is a generalization of a result by Russell Merris [3]. His result may be stated in the following way. Let $\chi$ be an irreducible character of $G$, let $M$ be the matrix obtained by applying $\chi$ to the entries of the multiplication table of $G$, let $A$ be any representation of $G$ affording $\chi$, and let $S$ be a subset of $G$. Then $\{A(g) \mid g \in S\}$ is linearly independent if and only if the rows of $M$ corresponding to $S$ are linearly independent. Our result strengthens Merris' result in three ways: (1) the condition about irreducibility is removed, (2) a way to determine the coefficients of any relation is given, and (3) smaller matrices are involved.

*Proof of Theorem.* Let $\chi_1, \cdots, \chi_r$ be the irreducible characters of $G$, and let $CG$ denote the complex group algebra of $G$. For each $k = 1, \cdots, r$, let

$$c_k = (\chi_k(e)/|G|) \sum_{g\in G} \chi_k(g) g \;.$$

Then $c_k$ is a central idempotent of $CG$ and corresponds to a representation of $G$ with character $\chi_k$ in the following way:

Let $R_k$ denote the principal ideal of $CG$ generated by $c_k$, and let $Z_k$ be any minimal left ideal of $CG$ contained in $R_k$. Then