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SCHUR'S THEOREM AND THE DRAZIN INVERSE

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It is shown that if $M = \begin{bmatrix} A & C \\ B & D \end{bmatrix}$ is a square $2n \times 2n$ matrix over a ring R, such that $AC = CA \in R_{n\times n}$, and with the property that A and C possess Drazin inverses, then M is invertible in $R_{2n\times 2n}$ if and only if DA-BC is invertible in $R_{n\times n}$.

1. Introduction. In a recent paper [7], Herstein and Small extended the classic result of Schur [5, p. 46] to matrices over E-rings. These are rings for which every primitive image is artinian. This result states that for a square complex block matrix $M = \begin{bmatrix} A & C \\ B & D \end{bmatrix}$, with A, B, C, D square of the same size such that $AC = CA$, then M is invertible exactly when $\Delta = DA - BC$ in invertible. This is a different but equivalent formulation of the problem as stated in [7].

The purpose of this note is to show that this result by Schur is basically a consequence of the *local* existence of the Drazin inverse [2] of the matrices A and C; that is, the strong-π-regularity of A and C [1] [4]. The proof of [7] was based on the fact that Schur's result for matrices over E-rings is really equivalent to the corresponding result for matrices over simple artinian rings (which may be taken to be division rings). Since artinian rings with unity are noetherian [8], p. 69, it follows that artinian rings with unity are strongly-π-regular, so that our local result extends the Schur theorem for artinian rings as proven in [7].

The Drazin inverse a^d of a ring element a, is the unique solution, if any, to the equations

$$a^k xa = a^k,\ xax = x,\ ax = xa\ , \tag{1}$$

for some $k \geqq 0$, while the group inverse $a^\#$ of a is the unique solution, if any, of these equations with $k = 0$, or 1. For example, if a is algebraic over some field $\mathscr{F}$ and $a^{n+1}b = a^n$, with $ab = ba$, then $a^d = a^n b^{n+1}$. The element a^d exists if and only if a is strongly-π-regular, that is, when both chains $\{a^iR\}$ and $\{Ra^i\}$ are ultimately stationary, [5, Theorem 4]. A ring element is called (von Neumann) *regular* if $aa^-a = a$ for some ring element a^-. If there exists such a^- that is invertible, a is called *unit*-regular.

We shall assume familiarity with the properties of these inverses [4] [2] [6] and in particular with the fact that $ac = ca \Rightarrow a^dc = ca^d$ [4, Theorem 1].

It is known that, unlike regularity and unit regularity, $R_{2\times 2}$ does