PACIFIC JOURNAL OF MATHEMATICS
Vol. 78, No. 1, 1978

OPERATOR CALCULUS

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To an analytic function $L(z)$ we associate the differential operator $L(D)$, D denoting differentiation with respect to a real variable x. We interpret L as the generator of a process with independent increments having exponential martingale $m(x(t), t) = \exp(zx(t) - tL(z))$. Observing that $m(x, -t) = e^{zC}1$ where $C = e^{tL}xe^{-tL}$, we study the operator calculus for C and an associated generalization of the operator xD, $A = CD$. We find what functions f have the property that $u_n = C^n f$ satisfy the evolution equation $u_t = Lu$ and the eigenvalue equations $Au_n = nu_n$, thus generalizing the powers x^n. We consider processes on R^N as well as R^1 and discuss various examples and extensions of the theory.

In the case that L generates a Markov semigroup, we have transparent probabilistic interpretations. In case L may not generate a probability semigroup, the general theory gives some insight into what properties any associated "processes with independent increments" should have. That is, the purpose is to elucidate the Markov case but in such a way that hopefully will lead to practicable definitions and will present useful ideas for defining more general processes—involving, say, signed and/or singular measures.

II. **Probabilistic basis.** Let $p_t(x)$ be the transition kernel for a process $\rho(t)$ with stationary independent increments. That is,

$$\int_A p_t(x) = \operatorname{Prob}(\rho(t) \in A \mid \rho(0) = 0) .$$

The Lévy-Khinchine formula says that, generally:

$$\int_R e^{i\xi x} p_t(x) = e^{tL(i\xi)}$$

where $L(i\xi) = ai\xi - \sigma^2\xi^2/2 + \int_{R-\{0\}} e^{i\xi u} - 1 - i\xi\eta(u) \cdot M(du)$ with

$$\eta(u) = u(|u| \leqq 1) + \operatorname{sgn} u(|u| \geqq 1) \quad \text{and} \quad \int \frac{u^2}{1+u^2} M(du) < \infty .$$

Denoting d/dx by D, this states that $L(D)$ is the generator of the process $\rho(t)$. It follows that

$$m(t) = e^{z\rho(t) - tL(z)} \text{ is a martingale}$$